11D Supergravity and Hidden Symmetries

11 维超引力与隐藏对称性

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Abstract

摘要

We review the structure of maximal $D=11$ and $D=10$ supergravities. Upon dimensional reduction, these theories give rise to the unique maximal supergravities in all lower spacetime dimensions $D<10$. In D dimensions, maximal supergravity exhibits the exceptional global symmetry group E_{11-D} , part of which is realized as hidden symmetries and only manifest after proper dualization of the fields. We also briefly review the reformulation of $D=11$ supergravity as an exceptional field theory, which renders the appearance of hidden symmetries manifest.

我们综述了极大 $D = 11$ 和 $D = 10$ 超引力的结构。经过维数约化，这些理论给出了所有更低时空维度 $D < 10$ 中唯一的极大超引力。在 D 维度中，极大超引力具有例外整体对称群 E_{11-D}，其中一部分作为隐藏对称性实现，仅在场经过适当对偶化后才显现。我们也简要综述了将 $D = 11$ 超引力重新表述为例外场论的工作，该表述令隐藏对称性的出现变得明确。

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超对称-超引力-高维-紧化-卡鲁扎-克莱因理论-规范对称性-隐藏对称性-例外群-例外场论

Introduction

引言

The interest to study supergravity theories in higher dimensions $D > 4$ and their dimensional reduction is (at least) twofold. On the one hand, supergravity theories appear as the low-energy effective action for string theories, which generically live in higher dimensions. On the other hand, from a purely four-dimensional point of view, the dimensional reduction of higher-dimensional supergravities can be considered as a powerful technique in order to construct extended $D = 4$ supergravity theories, i.e., supergravities with $\mathcal{N} > 1$ supercharges.

研究高维 $D > 4$ 超引力理论及其维数约化的兴趣至少有两方面来源。一方面，超引力理论会作为弦理论的低能有效作用量出现，而弦理论一般定义在高维空间中。另一方面，从纯粹四维的视角来看，高维超引力的维数约化是构造扩展 $D = 4$ 超引力理论 (即含有 $\mathcal{N} > 1$ 超荷的超引力) 的有力方法。

Under dimensional reduction of a minimally supersymmetric theory (i.e., a theory with a single supercharge), the spinor supercharge in general breaks into a number of lower-dimensional spinors, according to the fundamental spinor representations of the corresponding Poincaré groups (To get the correct counting, the reality conditions of spinors in the different dimensions of spacetime have to be taken into account (see, e.g., [1]).). Minimal supersymmetry in higher dimensions thus induces extended supersymmetry in the reduced theory. For globally supersymmetric theories, this led to the direct construction of maximally supersymmetric $D=4, \mathcal{N}=4$ Yang-Mills theory upon dimensional reduction of the ten-dimensional minimal $\mathcal{N}=1$ super Yang-Mills theory [2].

对于最小超对称理论 (即仅含一个超荷的理论) 进行维数约化时，自旋超荷根据相应庞加莱群的基本旋量表示，一般会分解为多个低维旋量 (为得到正确的计数，必须考虑不同时空维度下旋量的实性条件，参见例如文献 [1])。因此高维的最小超对称会在约化后的理论中诱导出扩展超对称。对于整体超对称理论，这一思路通过对十维最小 $\mathcal{N}=1$ 超杨-米尔斯理论做维数约化，直接构造出了最大超对称 $D=4, \mathcal{N}=4$ 杨-米尔斯理论 [2]。

For supergravity, eleven is the highest dimension in which the minimal super-symmetric extension of the Poincaré algebra allows for a supermultiplet of fields of spin smaller or equal to 2 [3]. For fields with spin larger than 2, interacting theories with a finite number of fields in general do not exist. Eleven is thus the highest dimension in which supergravity can be constructed. The associated interacting theory is unique and has been found by Cremmer, Julia, and Scherk [4]. Upon dimensional reduction to $D=4$ dimensions, this theory gives rise to maximally supersymmetric $\mathcal{N}=8$ supergravity [5, 6] . This is arguably the most remarkable extension of $D=4$ Einstein gravity due to its high degree of symmetry and the finiteness properties of its higher loop amplitudes [7-14].

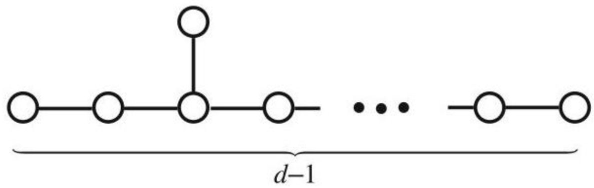
对于超引力，11 维是庞加莱代数的最小超对称扩展可以存在自旋不大于 2 的场组成超多重态的最高维度 [3]。对于自旋大于 2 的场，一般不存在含有限个场的相互作用理论。因此 11 维是可以构造超引力的最高维度，对应的相互作用理论是唯一的，由 Cremmer、Julia 和 Scherk 发现 [4]。将该理论维数约化到 $D=4$ 维后，就得到了最大超对称 $\mathcal{N}=8$ 超引力 [5, 6] 。由于其极高的对称性以及高阶圈振幅的有限性性质 [7-14]，该理论可以说是 $D=4$ 爱因斯坦引力最引人注目的推广。

The detour via eleven dimensions has in fact been an indispensable tool in the construction of this theory [5,6], in particular in order to determine the complicated nonlinear interactions between its 70 scalar fields, which had proven an extremely challenging task within the $D=4$ perturbative approach [15]. The other key ingredient in the construction of $\mathcal{N}=8$ supergravity has been the seminal observation that the theory admits an unexpectedly large global symmetry group, the exceptional group $\mathrm{E}_{7(7)}$ [5, 6] (The subscript in parentheses in this notation specifies the particular real form of the group: for $\mathrm{E}_{7(7)}$, the associated Lie algebra has 70 non-compact generators and 63 compact generators with the latter spanning the compact Lie algebra $\mathfrak{su}(8)$.). While part of these global symmetries can be understood from the gauge symmetries of the theory's higher-dimensional ancestor, a considerable part of the exceptional symmetries has no direct higher-dimensional interpretation and is often referred to as hidden symmetries. Their presence is essential for the realization of the exceptional symmetry group, which in turn allows to organize the couplings of the theory in a remarkably compact way, e.g., the scalar sector of the theory is most concisely described as a nonlinear sigma model on the coset space $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$.

实际上，绕道 11 维一直是构造该理论必不可少的工具 [5,6]，尤其是它帮助确定了理论中 70 个标量场之间复杂的非线性相互作用——这在 $D = 4$ 微扰框架下一直是极具挑战性的任务 [15]。构造 $\mathcal{N} = 8$ 超引力的另一关键要素是一项开创性发现: 该理论拥有一个出乎意料的大整体对称群，即例外群 $\mathrm{E}_{7(7)}$ [5, 6] (该记号中括号内的下标指定了群的特定实形式: 对于 $\mathrm{E}_{7(7)}$ ，其对应李代数有 70 个非紧生成元与 63 个紧生成元，其中紧生成元张成紧李代数 $\mathfrak{su}(8)$)。虽然这些整体对称性中有一部分可以从高维原理论的规范对称性得到解释，但相当一部分例外对称性没有直接的高维诠释，常被称为隐藏对称性。它们的存在对于例外对称群的实现至关重要，而后者反过来可以让理论的耦合以极为简洁的方式组织起来: 例如该理论的标量区可以最简洁地描述为陪集空间 $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ 上的非线性 sigma 模型。

Fig. 1 The Dynkin diagram of the exceptional Lie algebra $\mathfrak{e}_d$

图 1 例外李代数 $\mathfrak{e}_d$ 的 Dynkin 图



Subsequently, the presence of hidden symmetries and the appearance of exceptional global symmetry groups in maximal supergravity were recognized as part of a general pattern that has been dubbed the silver rules of supergravity [16-19]. Maximal supergravity in $D = 11 - d$ dimensions exhibits a global symmetry group $\mathrm{E}_{d(d)}$, realizing the series of exceptional Lie groups in the Dynkin classification, with the Dynkin diagram of the associated algebra given in Fig. 1. For small values of d , the exceptional series degenerates into the classical Lie groups

此后，人们认识到最大超引力中隐藏对称性的存在以及例外整体对称群的出现是一般模式的一部分，该模式被称为超引力的银规则 [16-19]。$D = 11 - d$ 维的最大超引力具有整体对称群 $\mathrm{E}_{d(d)}$ ，对应 Dynkin 分类中的例外李群序列，对应李代数的 Dynkin 图如图 1 所示。当 d 取较小值时，该例外序列退化为经典李群

$$\mathrm{E}_{5(5)} \simeq \mathrm{SO}(5,5)\,,\ \mathrm{E}_{4(4)} \simeq \mathrm{SL}(5)\,,\ \mathrm{E}_{3(3)} \simeq \mathrm{SL}(3) \times \mathrm{SL}(2)\,, \tag{1}$$

as can be found from properly extrapolating the general Dynkin diagram. Let us also note that discrete versions $\mathrm{E}_{d(d)}(\mathbb{Z})$ of the exceptional symmetry groups of supergravity survive in the toroidal compactification of the full string theories [20].

这可以通过对一般狄金图进行适当外推得到。我们还需要注意，超引力例外对称群的离散版本 $\mathrm{E}_{d(d)}(\mathbb{Z})$ 在完整弦理论的环面紧致化中得以保留 [20]。

In this chapter, we review the structure of maximal supergravity in eleven dimensions, its dimensional reduction, and the appearance of hidden symmetries. To this end, we first review in section ”Maximal Supergravity in $D = 11$ and $D = 10$ Dimensions” in some detail the field content and dynamics of the maximal supergravities in ten and eleven dimensions. In section ”Toroidal Reduction”, we discuss the toroidal compactification of these theories to lower-dimensional maximal supergravities. In particular, we determine the geometric symmetries of the lower-dimensional theories, i.e., the global symmetries that descend from particular diffeomorphism and gauge transformations in higher dimensions. Section ”Hidden Symmetries” then reviews the appearance of hidden global symmetries in lower dimensions. The central example is $D = 4, \mathcal{N} = 8$ supergravity with its global symmetry group enhanced to the full exceptional group $\mathrm{E}_{7(7)}$. Finally, in section ”Exceptional Field Theory”, we review the formulation of 11D supergravity as an exceptional field theory [21], which highlights the role of the full exceptional group in the full 11D supergravity before dimensional reduction.

本章我们将回顾十一维最大超引力的结构、它的维数约化，以及隐藏对称性的出现。为此，我们首先在“$D = 11$ 维和 $D = 10$ 维最大超引力”一节中详细回顾十维和十一维最大超引力的场内容与动力学。在“环面约化”一节中，我们讨论这些理论向低维最大超引力的环面紧致化。我们尤其确定了低维理论的几何对称性，即源自高维特定微分同胚和规范变换的整体对称性。随后“隐藏对称性”一节回顾了低维中隐藏整体对称性的出现。核心例子是 $D = 4, \mathcal{N} = 8$ 超引力，其整体对称群增强为完整例外群 $\mathrm{E}_{7(7)}$ 。最后，在“例外场论”一节中，我们回顾将 11 维超引力构造为例外场论的表述 [21]，该表述突出了完整例外群在维数约化前完整 11 维超引力中的作用。

Throughout this chapter, our discussion of hidden symmetries will mostly be restricted to the bosonic sectors of the supergravity theories. Although the fermionic field content and couplings are at the very origin of all these theories, the symmetry enhancement and the appearance of the exceptional symmetry groups can be realized and studied entirely within their bosonic sectors. In particular, even in the presence of fermions, the exceptional global symmetry algebras do not extend to larger superalgebras.

在整章中，我们对隐藏对称性的讨论大多仅限于超引力理论的玻色子部分。尽管费米子场内容和相互作用是所有这些理论的本源，对称性增强和例外对称群的出现都可以完全在玻色子部分内实现和研究。具体而言，即使存在费米子，例外整体对称代数也不会扩展为更大的超代数。

Maximal Supergravity in $D = 11$ and $D = 10$ Dimensions

$D = 11$ 维和 $D = 10$ 维最大超引力

Field Content

场内容

The highest-dimensional supergravity theory lives in eleven spacetime dimensions and was constructed by Cremmer, Julia, and Scherk [4]. Its field content is given by the lowest massless representation of the supersymmetry algebra

最高维超引力理论存在于 11 维时空，由 Cremmer、Julia 和 Scherk 构造 [4]。其场内容由超对称代数的最低质量表示给出

$$\{Q_\alpha, Q_\beta\} = 2P_M(\Gamma^M \Gamma^0)_{\alpha\beta}, \tag{2}$$

where Q_α are the 32 independent real supercharges in eleven-dimensional Minkowski spacetime. P_M with $M = 0, 1, \dots, 10$, is the $D = 11$ momentum, and Γ^M are the $\mathfrak{so}(1, 10)$ gamma matrices. For massless states which fulfill $P_M P^M = 0$, the r.h.s. of (2) describes a projector of half-maximal rank in spinor space. It follows that only 16 out of the 32 supercharges act nontrivially on massless states and satisfy the Clifford algebra $\mathcal{C}_{16}$, which admits a $2^8 = 256$ -dimensional irreducible representation. The spacetime interpretation of these states is inferred from the embedding of the massless little group

其中 Q_α 是 11 维闵氏时空里 32 个独立的实超荷。P_M 中 $M = 0, 1, \dots, 10$ 是 $D = 11$ 动量，Γ^M 是 $\mathfrak{so}(1, 10)$ 伽马矩阵。对于满足 $P_M P^M = 0$ 的无质量态，(2) 的右侧描述了旋量空间中一个半最大秩的投影算子。由此可知，32 个超荷中只有 16 个对无质量态非平凡作用，且满足 Clifford 代数 $\mathcal{C}_{16}$ ，该代数容许一个 $2^8 = 256$ 维不可约表示。我们可以通过嵌入无质量小群推导这些态的时空诠释

$$\mathcal{C}_{16} \supset \mathfrak{so}(16) \supset \mathfrak{so}(9), \tag{3}$$

according to

根据

$$256 \to 128_s + 128_c \to 44 + 84 + 128 \tag{4}$$

The 44 corresponds to the symmetric traceless product of two vectors: these are the degrees of freedom of a massless spin-2 field, the graviton G_{MN} . The $84 = \begin{pmatrix} 9 \\ 3 \end{pmatrix}$ on the other hand counts the degrees of freedom a totally antisymmetric massless 3-form field C_{KMN} , i.e., a field with local tensor gauge symmetry

其中 44 对应两个矢量的对称无迹乘积: 这对应无质量自旋 2 场即引力子 G_{MN} 的自由度。而 $84 = \begin{pmatrix} 9 \\ 3 \end{pmatrix}$ 对应全反对称无质量 3 形式场 C_{KMN} 的自由度，即一个具有局域张量规范对称性的场

$$\delta C_{KMN} = 3\partial_{[K}\Lambda_{MN]}. \tag{5}$$

The 128 finally corresponds to the degrees of freedom of a massless spin-3/2 field in eleven dimensions, the gravitino Ψ_M . The full 11D supergravity multiplet thus is given by

最后 128 对应 11 维下无质量自旋 3/2 场即引力微子 Ψ_M 的自由度。因此完整的 11 维超引力多重态由下式给出

$$\{G_{MN}, \psi_M, C_{KMN}\} . \tag{6}$$

The interacting theory has been constructed in [4] and is reviewed in section "11D Supergravity". The same reasoning shows why supergravity theories do not exist beyond eleven dimensions: repeating the above analysis for, say, a twelve-dimensional spacetime with 64 supercharges yields a minimal field content that includes fields with spin larger than two. No consistent interacting theory for such fields can be constructed (unless infinitely many fields are included).

相互作用理论已在文献 [4] 中构造，本文将在“11 维超引力”小节回顾。同样的推导可以解释为何超引力理论不存在于 11 维以上: 对 12 维时空、64 个超荷重复上述分析，得到的最小场内容会包含自旋大于 2 的场。除非引入无穷多个场，否则这类场无法构造出自洽的相互作用理论。

A similar analysis yields the field content of $D = 10$ supergravity. In this case, the smallest massless representation descends from the Clifford algebra $\mathcal{C}_8$ and comprises $2^4 = 16$ states. In analogy to (4), they transform as vector and spinor under the little group SO(8)

类似的分析可以给出 $D = 10$ 超引力的场内容。此时，最小无质量表示从 Clifford 代数 $\mathcal{C}_8$ 导出，包含 $2^4 = 16$ 个态。和 (4) 类似，这些态在小群 SO(8) 下按矢量和旋量变换

$$16 \to 8v + 8s \tag{7}$$

counting degrees of freedom of a ten-dimensional massless vector and a matter fermion. This is the minimal $\mathcal{N} = 1$ vector multiplet in ten dimensions. The higher massless multiplets can be found by tensoring (7) with the fundamental representations of SO (8) . This results in the $\mathcal{N} = 1$ supergravity multiplet and two inequivalent gravitino multiplets:

对应 10 维无质量矢量和物质费米子的自由度计数。这就是 10 维中最小的 $\mathcal{N} = 1$ 矢量多重态。更高阶的无质量多重态可以通过将 (7) 与 SO (8) 的基础表示做张量积得到，由此得到 $\mathcal{N} = 1$ 超引力多重态和两个不等价的引力微子多重态:

$$\text{sugra}: 8_v \otimes (8_v + 8_s) = 1 + 28 + 35_v + 8_c + 56_c,$$

$$\text{gravitino A}: 8_c \otimes (8_v + 8_s) = 8_v + 56_v + 8_s + 56_s\,,$$

$$\text{gravitino B}: 8_s \otimes (8_v + 8_s) = 1 + 28 + 35_s + 8_c + 56_c\,. \tag{8}$$

Similar to (4), the physical field content of these multiplets may be inferred from the representations of the little group. The $\mathcal{N} = 1$ supergravity multiplet carries the metric, a scalar field (the dilaton), and an antisymmetric 2-form together with a gravitino and a matter fermion. The interacting theory exists, and upon coupling to the vector multiplet (7) in the adjoint representation of the gauge group, this describes the low-energy effective theory of the heterotic string [22].

和 (4) 类似，我们可以从小群的表示推导出这些多重态的物理场内容。$\mathcal{N} = 1$ 超引力多重态包含度规、一个标量场 (即 dilaton，dilaton)、一个反对称 2 形式，以及引力微子和一个物质费米子。该相互作用理论自洽存在；当它耦合到规范群伴随表示的矢量多重态 (7) 时，恰好描述杂弦的低能有效理论 [22]。

The first gravitino multiplet in (8) carries a spacetime vector and a 3-form together with a gravitino and a matter fermion of chirality opposite to the fermions of the supergravity multiplet. Instead, the second gravitino multiplet carries another scalar (the axion) and a 2-form together with a self-dual 4-form (the 35_s in (8)) (see

(30) below). Its fermions are a gravitino and a matter fermion of the same chirality as the fermions of the supergravity multiplet. Coupling of a massless gravitino multiplet requires supersymmetry enhancement. The two resulting theories exist as maximally supersymmetric interacting theories and are denoted as the non-chiral $\mathcal{N} = (1,1)$ type IIA and the chiral $\mathcal{N} = (2,0)$ type IIB theory, respectively. They are the low-energy effective theories for the massless spectrum of IIA and IIB strings, respectively. The type IIA theory can be obtained by compactifying 11D supergravity on a circle S^1 as will be discussed in section ”IIA Supergravity”. In contrast, the type IIB supergravity does not have a higher-dimensional origin and has been constructed in [23-25]. We will review it in section ”IIB Supergravity”.

(8) 式中第一个引力微子多重态包含一个时空矢量、一个 3-形式，以及一个引力微子和一个手征性与超引力多重态费米子相反的物质费米子。第二个引力微子多重态则包含另一个标量 (轴子)、一个 2-形式，以及一个自对偶 4-形式 (即 (8) 式中的 35_s ，见下文 (30) 式)。其费米子为一个引力微子和一个物质费米子，手征性与超引力多重态费米子相同。无质量引力微子多重态的耦合要求超对称增强。由此得到的两种理论都是极大超对称相互作用理论，分别记为非手征的 $\mathcal{N} = (1,1)$ IIA 型和手征的 $\mathcal{N} = (2,0)$ IIB 型理论。它们分别是 IIA 弦和 IIB 弦无质量谱的低能有效理论。如“IIA 超引力”章节所述，IIA 理论可以通过 11 维超引力在圆 S^1 上紧致化得到。与之相对，IIB 超引力没有更高维起源，已在文献 [23-25] 中构造，我们将在“IIB 超引力”章节对其进行回顾。

11D Supergravity

11 维超引力

The action of eleven-dimensional supergravity with field content given in (6) has been constructed in [4]. To quadratic order in the fermions, its Lagrangian extends the standard combination of Einstein-Hilbert and Rarita-Schwinger term

场内容由 (6) 给出的十一维超引力的作用量已在文献 [4] 中构造完成。在费米子二次阶范围内，其拉格朗日量拓展了爱因斯坦-希尔伯特项与拉里塔-施温格项的标准组合

$$\mathcal{L}_0\,[E,\psi] = |E|\,R\,[\omega] - \frac{1}{2}\,|E|\,\overline{\psi}_K\Gamma^{KMN}D[\omega]_M\psi_N, \tag{9}$$

by a kinetic, a topological, and a fermionic interaction term for the antisymmetric 3-form field C_{KMN} , given by

，新增了反对称 3-形式场 C_{KMN} 的动力学项、拓扑项和费米相互作用项，形式如下

$$\mathcal{L}_C\,[E,C,\psi] = -\frac{1}{48}\,|E|\,F_{KLMN}F^{KLMN}$$

$$+\frac{1}{144^2}\varepsilon^{N_0N_1\ldots N_{10}}F_{N_0\ldots N_3}F_{N_4\ldots N_7}C_{N_8N_9N_{10}}$$

$$+\frac{1}{192}\,|E|\left(\overline{\psi}_P\Gamma^{KLMNPQ}\psi_Q + 12\overline{\psi}^K\Gamma^{LM}\psi^N\right)F_{KLMN}.$$

(10)

Here, $|E|$ denotes the determinant of the eleven-bein $E_M{}^A$, related to the metric as

式中 $|E|$ 表示十一 vielbein(十一维标架) $E_M{}^A$ 的行列式，它与度规的关系为

$$G_{MN} = E_M{}^A E_{NA}, \tag{11}$$

with flat Lorentz indices A (Throughout this chapter, we use spacetime signature $(-++\ldots+)$.). The gauge invariant abelian field strength is defined as $F_{KLMN} = 4\partial_{[K}C_{LMN]}$. The totally antisymmetric (numerical) $\varepsilon^{N_0N_1\ldots N_{10}}$ is the Levi-Civita density. The Chern-Simons term $F\wedge F\wedge C$ is invariant under tensor gauge transformations (5) up to a total derivative. The appearance of such topological terms is a generic feature in higher-dimensional supergravity theories. The full 11D supergravity Lagrangian is given by

，其中 A 为平坦洛伦兹指标 (本章通篇我们使用时空号差 $(-++\ldots+)$ 。)。规范不变的阿贝尔场强定义为 $F_{KLMN} = 4\partial_{[K}C_{LMN]}$ 。完全反对称 (数值) 张量 $\varepsilon^{N_0N_1\ldots N_{10}}$ 是列维-奇维塔密度。陈-西蒙斯项 $F\wedge F\wedge C$ 在张量规范变换 (5) 下，只差一个全导数保持不变。这类拓扑项是高维超引力理论的普遍特征。完整的 11 维超引力拉格朗日量为

$$\mathcal{L}_{11\mathrm{D}} = \mathcal{L}_0 + \mathcal{L}_C + \mathcal{L}_{\psi^4}, \tag{12}$$

where the quartic fermion terms of $\mathcal{L}_{\psi^4}$ can be formally absorbed into an appropriate modification of the spin connection ω and the 4-form field strength im the ψ^2 terms in (9), (10). The full Lagrangian (12) is invariant under the supersymmetry transformations

，式中 (12)(10) 里 $\mathcal{L}_{\psi^4}$ 的四次费米子项可以通过适当修正自旋联络 ω 和 4-形式场强，形式上被吸收进 ψ^2 项中。完整拉格朗日量 (12) 在如下超对称变换下保持不变

$$\delta E_M{}^A = \frac{1}{4}\bar{\varepsilon}\Gamma^A\psi_M,\ \delta C_{KLM} = \frac{3}{4}\bar{\varepsilon}\Gamma_{[KL}\psi_{M]},$$

$$\delta\psi_M = D[\omega]_M\varepsilon - \frac{1}{288}F^{KLPQ}\Gamma_{MKLPQ}\varepsilon + \frac{1}{36}F_{MNPQ}\Gamma^{NPQ}\varepsilon, \tag{13}$$

given up to cubic terms in the fermions. In turn, supersymmetry uniquely fixes all the terms in (12) and in particular requires the presence of the topological Chern-Simons term in (10).

，该变换仅给出到费米子三次项。反过来，超对称唯一确定了 (12) 中的所有项，尤其要求了 (10) 中拓扑陈-西蒙斯项的存在。

Let us still point out two interesting properties of the Lagrangian (12), which will be important for the appearance of hidden symmetries. First, the field equations descending from the Lagrangian (12), scale homogeneously under the following transformation:

下面我们指出拉格朗日量 (12) 的两个有趣性质，它们对隐藏对称性的出现非常重要。首先，由拉格朗日量 (12) 导出的场方程，在如下变换下齐次缩放:

$$\delta E_M{}^A = \zeta E_M{}^A,\ \delta C_{KMN} = 3\zeta C_{KMN},\ \delta\psi_M = \frac{\zeta}{2}\psi_M, \tag{14}$$

with constant ζ . In general spacetime dimension D , this trombone symmetry is not a symmetry of the action but rescales the Lagrangian as

，其中 ζ 是常数。对于一般时空维数 D ，这种长号对称性并不是作用量的对称性，它仅对拉格朗日量做如下缩放:

$$\delta\mathcal{L} = (D-2)\zeta\mathcal{L}. \tag{15}$$

It still plays an important role, e.g., among the spectrum-generating symmetries for the fundamental BPS solutions [26]. The trombone symmetry is present in all two-derivative supergravity theories but is in general broken by higher-order corrections.

但它仍然发挥重要作用，例如在基础 BPS 解的谱生成对称性中就是如此 [26]。长号对称性存在于所有二阶导数超引力理论中，但一般会被高阶修正破坏。

Second, the field equations for the 3-form C_{KLM} may be used as integrability relations, which ensure the consistency of the definition of an antisymmetric 6-form potential $C_{N_1\ldots N_6}$ by means of the first-order duality equation

其次，3-形式 C_{KLM} 的场方程可作为可积性关系，它保证了我们可以通过一阶对偶方程定义反对称 6-形式势 $C_{N_1\ldots N_6}$ ，且该定义是自洽的:

$$F_{N_1N_2\ldots N_7} = -\frac{1}{24}\,|E|\,\varepsilon_{N_1N_2\ldots N_7M_1M_2M_3M_4}F^{M_1M_2M_3M_4} + \text{ fermions}\,. \tag{16}$$

Here, the 7-form field strength is defined as

式中，7-形式场强定义为

$$F_{N_1N_2\ldots N_7} = 7\partial_{[N_1}C_{N_2N_3\ldots N_7]} + \frac{35}{2}C_{[N_1N_2N_3}F_{N_4N_5N_6N_7]}, \tag{17}$$

with a nontrivial Bianchi identity

，满足非平凡比安基恒等式

$$8\partial_{[N_1}F_{N_2N_3\ldots N_8]} = 35F_{[N_1N_2N_3N_4}F_{N_5N_6N_7N_8]}. \tag{18}$$

Indeed, hitting (16) with another external derivative and using (18), this equation reduces to the field equations for the 3-form C_{KMN} , obtained by variation of (10). In particular, the contribution from the Chern-Simons term in (10) requires the second term in (17). As a consequence, the 6-form $C_{N_1\ldots N_6}$ transforms nontrivially under the gauge transformations (5). Specifically, its field strength (17) is invariant under the gauge transformations

事实上，对 (16) 作用一次外导数再代入 (18)，该式就约化为对 (10) 做变分得到的 3-形式 C_{KMN} 场方程。特别地，(10) 中陈-西蒙斯项的贡献要求了 (17) 中的第二项存在。因此，6-形式 $C_{N_1\ldots N_6}$ 在规范变换 (5) 下会发生非平凡变换。具体来说，它的场强 (17) 在如下规范变换下保持不变

$$\delta C_{N_1 N_2 \ldots N_6} = 6\partial_{[N_1} \Lambda_{N_2 N_3 \ldots N_6]-30} C_{[N_1 N_2 N_3} \partial_{N_4} \Lambda_{N_5 N_6]}, \tag{19}$$

with a new gauge parameter $\Lambda_{N_1 \ldots N_5}$, and Λ_{MN} from (5). This shows that the full algebra of 11D gauge symmetries is actually non-Abelian

，其中 $\Lambda_{N_1 \ldots N_5}$ 是新的规范参数，Λ_{MN} 取自 (5)。这说明 11 维规范对称的完整代数实际上是非阿贝尔的

$$\left[\delta_{\Lambda_1}, \delta_{\Lambda_2}\right] = \delta_{\Lambda_{12}}, \tag{20}$$

with the commutator of two 3-form gauge transformations (5), (19), resulting in a 6-form gauge transformation with parameter

，两个 3-形式规范变换 (5)、(19) 的对易子，会给出一个带有如下参数的 6-形式规范变换:

$$\Lambda_{12, N_1 N_2 \ldots N_5} = 15\Lambda_{2,[N_1 N_2]} \partial_{N_3} \Lambda_{1, N_4 N_5]} - 15\Lambda_{1,[N_1 N_2} \partial_{N_3} \Lambda_{2, N_4 N_5]}. \tag{21}$$

The introduction of dual fields, defined by first-order duality equations such as (16), is a general feature of supergravity theories. In general spacetime dimension D , the field equations for a given p -form allow the introduction of its dual form of degree $(D-p-2)$, in terms of which Bianchi identities and equations of motion become exchanged. As a consequence, supergravity theories may admit different Lagrangian formulations, which are on-shell equivalent only after relating their fields by means of duality equations such as (16) (Specifically, in dimensional reductions from 11D supergravity, the lower-dimensional duality equations are precisely obtained by dimensional reduction of (16).). In particular, there is always a Lagrangian formulation of the theory in which all forms are dualized to the lowest possible degree. It is in this form that the largest global symmetry group becomes visible, as we shall see below.

引入由一阶对偶方程 (如方程 (16)) 定义的对偶场，是超引力理论的普遍特征。在任意时空维度 D 中，给定 p -形式的场方程允许引入其次数为 $(D-p-2)$ 的对偶形式，在此变换下，比安基恒等式与运动方程会互换。因此，超引力理论可以存在不同的拉格朗日表述，这些表述只有通过对偶方程 (如 (16)) 关联场后，才是壳等价的 (具体来说，在 11 维超引力的维度约化中，低维对偶方程正是由 (16) 维度约化得到)。尤其值得注意的是，该理论始终存在一种拉格朗日表述，其中所有形式都被对偶化为最低可能次数。我们下文将看到，正是在这种形式下，最大整体对称群才会显现出来。

IIA Supergravity

IIA 超引力

Dimensional reduction of 11D supergravity on a circle S^1 yields maximal type IIA supergravity in ten dimensions [27, 28]. Explicitly, the reduction amounts to imposing independence of all fields on the eleventh coordinate

将 11 维超引力在圆 S^1 上做维约化，得到十维的极大 IIA 型超引力 [27, 28]。具体而言，约化相当于要求所有场都不依赖于第十一坐标

$$\partial_{10} G_{MN} = 0 = \partial_{10} C_{KMN}, \tag{22}$$

together with a standard Kaluza-Klein ansatz for the eleven-dimensional metric (Compared to the general Kaluza-Klein parametrization (45) given below, this ansatz uses a rescaled dilaton $\phi \to \frac{2}{3}\phi$ in order to match some standard conventions of IIA supergravity.)

并对十一维度量采用标准卡鲁扎-克莱因近似 (相较于下文给出的一般卡鲁扎-克莱因参数化 (45)，该近似使用缩并的 dilaton $\phi \to \frac{2}{3}\phi$，以匹配 IIA 超引力的部分标准约定。)

$$ds^2_{(11)} = \mathrm{e}^{-\phi/6} g_{\mu\upsilon} dx^\mu dx^\upsilon + \mathrm{e}^{4\phi/3} \left(dy + A_\mu dx^\mu\right)^2, \tag{23}$$

in terms of the ten-dimensional metric $g_{\mu\nu}$,(with indices $\mu = 0, \ldots, 9$, labelling the coordinates of the ten-dimensional spacetime), a dilaton ϕ , and a Kaluza-Klein vector A_μ . Similarly, the components of the eleven-dimensional 3-form are parametrized as

它可以用十维度量 $g_{\mu\nu}$ (其中指标 $\mu = 0, \ldots, 9$ 标记十维时空的坐标)、dilaton ϕ 和卡鲁扎-克莱因矢量 A_μ 表示。类似地，十一维 3-形式的分量可参数化为

$$C_{\mu\upsilon 10} = B_{\mu\upsilon},\ C_{\mu\upsilon\rho} = A_{\mu\upsilon\rho} + 3A_{[\mu} B_{\upsilon\rho]}, \tag{24}$$

in terms of a 2-form $B_{\mu\nu}$ and a 3-form $A_{\mu\nu\rho}$. Under the eleven-dimensional gauge transformations (5), these forms transform as

可以用 2-形式 $B_{\mu\nu}$ 和 3-形式 $A_{\mu\nu\rho}$ 表示。在十一维规范变换 (5) 下，这些形式变换为

$$\delta B_{\mu\nu} = 2\partial_{[\mu}\Lambda_{\nu]},\ \delta A_{\mu\nu\rho} = 3\partial_{[\mu}\Lambda_{\nu\rho]} - 6A_{[\mu}\partial_\nu\Lambda_{\rho]}, \tag{25}$$

respectively, where we denote $\Lambda_\mu = \Lambda_{\mu 10}$. Accordingly, their gauge invariant field strengths are defined as

分别为，这里我们记 $\Lambda_\mu = \Lambda_{\mu 10}$ 。相应地，它们的规范不变场强定义为

$$H_{\mu\upsilon\rho} = 3\partial_{[\mu} B_{\upsilon\rho]},\ F_{\mu\upsilon\rho\sigma} = 4\partial_{[\mu} A_{\upsilon\rho\sigma]} + 6F_{[\mu\upsilon} B_{\rho\sigma]}, \tag{26}$$

where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ denotes the Abelian field strength of the Kaluza-Klein vector from (23). Plugging the reduction ansatz (22), (23), (24), into the supergravity Lagrangian (12), yields the ten-dimensional Lagrangian of type IIA supergravity

其中 $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ 表示式 (23) 中卡鲁扎-克莱因矢量的阿贝尔场强。将约化近似 (22)、(23)、(24) 代入超引力拉格朗日量 (12)，得到 IIA 超引力的十维拉格朗日量

$$
\begin{aligned}
\mathcal{L}_{\mathrm{IIA}} = |e|\, R_{(10)} - \frac{1}{4}\, |e|\, \mathrm{e}^{3\phi/2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2}\, |e|\, \partial_\mu \phi \partial^\mu \phi \\
-\frac{1}{48}\, |e|\, \mathrm{e}^{\phi/2} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} - \frac{1}{12}\, |e|\, \mathrm{e}^{-\phi} H_{\mu\nu\rho} H^{\mu\nu\rho} \\
+\frac{1}{144^2} \varepsilon^{\mu_0\mu_1\ldots\mu_9} F_{\mu_0\mu_1\mu_2\mu_3} \left(3 F_{\mu_4\mu_5\mu_6\mu_7} B_{\mu_8\mu_9} - 8 H_{\mu_4\mu_5\mu_6} A_{\mu_7\mu_8\mu_9}\right) \\
+ \text{fermions}.
\end{aligned} \tag{27}
$$

Here, $|e|$ and $R_{(10)}$ are the determinant of the ten-dimensional vielbein and the ten-dimensional Ricci scalar, respectively. The topological term is invariant under gauge transformations (25) up to a total derivative. The field content of (27) matches the IIA supergravity multiplet of (8). In particular, reduction of the 11D gravitino gives rise to two ten-dimensional gravitini of opposite chirality. The IIA supergravity Lagrangian may be equivalently expressed in terms of different fundamental fields (e.g., using the original components $C_{\mu\nu\rho}$ rather than the $A_{\mu\nu\rho}$ of (24)). However, all formulations share the property that the field strength $F_{\mu\nu\rho\sigma}$ building the kinetic term for the 3-form satisfies a nontrivial Bianchi identity

此处，$|e|$ 和 $R_{(10)}$ 分别是十维标架的行列式和十维里奇标量。拓扑项在规范变换 (25) 下仅相差一个全导数，保持不变。式 (27) 的场内容与 (8) 的 IIA 超引力多重态一致。具体来说，11 维引力子的约化会产生两个手征相反的十维引力子。IIA 超引力拉格朗日量可以用不同的基本场等价表示 (例如，使用原始分量 $C_{\mu\nu\rho}$ 而非 (24) 的 $A_{\mu\nu\rho}$)。但所有表述都具有一个共同性质: 构造 3-形式动能项的场强 $F_{\mu\nu\rho\sigma}$ 满足非平凡比安基恒等式

$$
5\partial_{[\mu} F_{\nu\rho\sigma\tau]} = 10 F_{[\mu\nu} H_{\rho\sigma\tau]}. \tag{28}
$$

Let us also note that the Lagrangian (27) admits a 1-parameter massive deformation upon deforming the field strengths

我们还要指出，在对场强做形变后，拉格朗日量 (27) 允许一个单参数的质量形变

$$
F_{\mu\nu} \to F_{\mu\nu} + m B_{\mu\nu},\; F_{\mu\nu\rho\sigma} \to F_{\mu\nu\rho\sigma} + 3m B_{[\mu\nu} B_{\rho\sigma]}, \tag{29}
$$

thereby inducing a mass term for the 2-form $B_{\mu\nu}$ [29]. The resulting theory has an equivalent description in terms of a 9-form gauge potential, which reflects the presence of D8-branes in IIA string theory [30,31].

从而为 2-形式 $B_{\mu\nu}$ 诱导出一个质量项 [29]。该结果理论可以通过 9-形式规范势得到等价描述，这反映出 IIA 弦理论中存在 D8 膜 [30, 31]。

IIB Supergravity

IIB 型超引力

As shown in Equation (8), the bosonic field content of the ten-dimensional type IIB supergravity comprises the metric, two scalar fields, two 2-form gauge potentials $C_{\mu\nu}{}^{\alpha}, \alpha = 1, 2$, and a selfdual 4-form potential $C_{\mu\nu\rho\sigma}$. Specifically, the latter satisfies a first-order self-duality equation

如式 (8) 所示, 十维 IIB 型超引力的玻色子场内容包括度规、两个标量场、两个 2-形式规范势 $C_{\mu\nu}{}^{\alpha}, \alpha = 1, 2$ ， 以及一个自对偶 4-形式势 $C_{\mu\nu\rho\sigma}$ 。具体而言，后者满足一阶自对偶方程

$$F_{\mu\nu\rho\sigma\tau} = \frac{1}{5!} |e| \, \varepsilon_{\mu\nu\rho\sigma\tau\mu_1\mu_2\mu_3\mu_4\mu_5} F^{\mu_1\mu_2\mu_3\mu_4\mu_5}, \tag{30}$$

in terms of the field strength

用场强表示为

$$F_{\mu_1\ldots\mu_5} = 5\partial_{[\mu_1} C_{\mu_2\ldots\mu_5]} - \frac{5}{4} \varepsilon_{\alpha\beta} C_{[\mu_1\mu_2}{}^{\alpha} F_{\mu_3\mu_4\mu_5]}{}^{\beta}. \tag{31}$$

Here, $\varepsilon_{\alpha\beta}$ is the antisymmetric tensor in two indices, and

此处 $\varepsilon_{\alpha\beta}$ 是二阶反对称张量，且

$$F_{\mu\nu\rho}{}^{\alpha} = 3\partial_{[\mu} C_{\nu\rho]}{}^{\alpha} \tag{32}$$

is the Abelian field strength for the doublet of 2-forms. IIB supergravity has been constructed in [23-25]. Self-duality equations such as (30) cannot be derived from a standard action principle. As a consequence, these equations are often imposed separately, while the remaining field equations of IIB supergravity are most conveniently derived from a so-called pseudo-action with Lagrangian given by

是该 2-形 doublet 的阿贝尔场强。IIB 超引力由文献 [23-25] 构造完成。类似 (30) 的自对偶方程无法从标准作用量原理导出，因此这类方程通常需要单独额外添加；而 IIB 超引力其余的场方程可以很方便地从所谓的伪作用量得到，其拉格朗日量为

$$\begin{aligned}\mathcal{L}_{\mathrm{IIB}} = {} & |e| \, R_{(10)} + \frac{1}{4} |e| \, \partial_\mu m_{\alpha\beta} \partial^\mu m^{\alpha\beta} - \frac{1}{12} |e| \, F_{\mu_1\mu_2\mu_3}{}^{\alpha} F^{\mu_1\mu_2\mu_3\beta} m_{\alpha\beta} \\ & - \frac{1}{30} |e| \, F_{\mu_1\mu_2\mu_3\mu_4\mu_5} F^{\mu_1\mu_2\mu_3\mu_4\mu_5} \\ & - \frac{1}{864} \varepsilon_{\alpha\beta} \varepsilon^{\mu_1\ldots\mu_{10}} C_{\mu_1\mu_2\mu_3\mu_4} F_{\mu_6\mu_7\mu_8}{}^{\alpha} F_{\mu_8\mu_9\mu_{10}}{}^{\beta} \\ & + \text{fermions}.\end{aligned} \tag{33}$$

Here the matrix $m_{\alpha\beta}$ denotes a symmetric 2×2 matrix of unit determinant, which is parametrized by the two scalars, ϕ and C_0 , of the theory

此处矩阵 $m_{\alpha\beta}$ 表示单位行列式的对称 2×2 矩阵，由该理论的两个标量 ϕ 和 C_0 参数化

$$m_{\alpha\beta} = \frac{1}{\Im\tau}\begin{pmatrix} |\tau|^2 & -\Re\tau \\ -\Re\tau & 1 \end{pmatrix}_{\alpha\beta}, \ \tau = C_0 + i\mathrm{e}^{-\phi}. \tag{34}$$

Its inverse is denoted as $m^{\alpha\beta} = \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}m_{\gamma\delta}$, such that the kinetic term in (33) describes the sigma model on the coset space SL (2) /SO (2) , c.f. section "Coset Spaces",

其逆矩阵记为 $m^{\alpha\beta} = \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}m_{\gamma\delta}$ ，因此 (33) 中的动能项描述了陪集空间 SL (2) /SO (2) 上的 sigma 模型，参见“陪集空间”一节，

$$\frac{1}{4}\partial_\mu m_{\alpha\beta}\partial^\mu m^{\alpha\beta} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mathrm{e}^{2\phi}\partial_\mu C_0\partial^\mu C_0. \tag{35}$$

Type IIB supergravity is manifestly invariant under a global SL(2) symmetry, acting on all indices α, β , inducing a nonlinear action on the scalars ϕ, C_0 via (34). Furthermore, the Lagrangian (33) is invariant under gauge transformations

IIB 型超引力在整体 SL(2) 对称性下明显不变，该对称性作用在所有指标 α, β 上，通过 (34) 对标量 ϕ, C_0 诱导出非线性作用。此外，拉格朗日量 (33) 在规范变换下不变

$$\delta C_{\mu\upsilon}{}^{\alpha} = 2\partial_{[\mu}\Lambda_{\upsilon]}{}^{\alpha}$$

$$\delta C_{\mu\upsilon\rho\sigma} = 4\partial_{[\mu}\Lambda_{\upsilon\rho\sigma]} + \frac{1}{2}\varepsilon_{\alpha\beta}\Lambda_{[\mu}{}^{\alpha}F_{\upsilon\rho\sigma]}{}^{\beta}, \tag{36}$$

up to total derivatives from variation of the topological term. It is straightforward to verify that the integrability conditions of the self-duality equations (30) coincide with the second-order field equations obtained by variation of (33). Various alternative action principles for IIB supergravity have been put forward in the literature in order to also derive the self-duality equations (30) from a variational principle, typically at the expense of introducing additional fields and/or sacrificing manifest (9+1)-dimensional Lorentz invariance [32-38].

仅相差拓扑项变分产生的全导数。可以直接验证，自对偶方程 (30) 的可积性条件与对 (33) 变分得到的二阶场方程完全一致。文献中已经提出了多种 IIB 超引力的替代作用量原理，试图也从变分原理导出自对偶方程 (30)，但这通常需要引入额外场，和/或牺牲明显的 (9+1) 维洛伦兹不变性 [32-38]。

The SL(2) conventions of (33),(34) can be translated into the SU (1, 1) /U (1) conventions of [24] by combining the real components of the doublet $F_{\mu\upsilon\rho}{}^{\alpha}$ into a complex field strength

可以将 (33)、(34) 中的 SL(2) 约定转换为文献 [24] 中的 SU (1, 1) /U (1) 约定，方法是将 doublet $F_{\mu\upsilon\rho}{}^{\alpha}$ 的实分量组合为复场强

$$F_{\mu\nu\rho} \equiv F_{\mu\nu\rho}{}^{1} + iF_{\mu\nu\rho}{}^{2}, \tag{37}$$

and parametrizing the matrix $m_{\alpha\beta}$ in terms of a single complex scalar field B as

并将矩阵 $m_{\alpha\beta}$ 用单个复标量场 B 参数化为

$$m_{\alpha\beta} \equiv (1 - BB^*)^{-1} \begin{pmatrix} (1-B)(1-B^*) & i(B-B^*) \\ i(B-B^*) & (1+B)(1+B^*) \end{pmatrix}_{\alpha\beta}. \tag{38}$$

In terms of the complex combinations

用这些复组合表示

$$G_{\mu\upsilon\rho} \equiv (1 - BB^*)^{-1/2} \left(F_{\mu\upsilon\rho} - BF^*_{\mu\upsilon\rho}\right), \; P_\mu \equiv (1 - BB^*)^{-1} \partial_\mu B, \tag{39}$$

the kinetic terms of (33) translate into those of [24] with

(33) 中的动能项就转换为文献 [24] 中的形式，对应

$$m_{\alpha\beta} F_{\mu\upsilon\rho}{}^{\alpha} F^{\mu\upsilon\rho\beta} = G^*_{\mu\upsilon\rho} G^{\mu\upsilon\rho}, \; \frac{1}{4} \partial_\mu m_{\alpha\beta} \partial^\mu m^{\alpha\beta} = -2 P^*_\mu P^\mu. \tag{40}$$

Let us finally note that the Lagrangians (27), (33) of type IIA and type IIB supergravity coincide when truncated to the common field content $\{g_{\mu\nu}, \phi, B_{\mu\nu} = \; C_{\mu\upsilon}{}^1\}$, which is the bosonic part of the $\mathcal{N} = 1$ supergravity multiplet from (8) or the NS-NS sector.

最后我们注意到，截断到公共场内容 $\{g_{\mu\nu}, \phi, B_{\mu\nu} = \; C_{\mu\upsilon}{}^1\}$ 时，IIA 型和 IIB 型超引力的拉格朗日量 (27)、(33) 一致，该公共场内容就是 (8) 中给出的 $\mathcal{N} = 1$ 超引力多重态的玻色部分，即 NS-NS sector。

Toroidal Reduction

环面约化

Maximal supergravities can be obtained by Kaluza-Klein reduction from 11D and type IIB supergravity. In particular, the reduction of 11D supergravity on a d -dimensional torus T^d yields the maximal ungauged supergravity in $D = 11 - d$ dimensions. More precisely, for $D < 10$, this is the unique supergravity theory in D dimensions with 32 real supercharges and no fields charged under the Abelian gauge group.

最大超引力可通过 11 维超引力与 IIB 型超引力的卡鲁扎-克莱因约化得到。具体而言，11 维超引力在 d 维环面 T^d 上的约化，会得到 $D = 11 - d$ 维的最大无规范超引力。更准确地说，对 $D < 10$ 而言，这是 D 维中唯一满足以下条件的超引力理论: 拥有 32 个实超荷，且不存在阿贝尔规范群下带电的场。

In this section, we first discuss the global geometric GL (d) symmetry appearing after reduction to D dimensions as a remnant of the higher-dimensional diffeomor-phisms acting on the torus T^d . We explicitly perform the dimensional reduction on a torus, first for pure gravity and next for the p -forms, which typically span the matter sector of higher-dimensional supergravities. In section "Hidden Symmetries", we then discuss the enhancement of the geometric symmetry group by the so-called hidden symmetries.

在本节中，我们首先讨论约化到 D 维后出现的整体几何 $\mathrm{GL}((d))$ 对称性，它是作用在环面 T^d 上的高维微分同胚的剩余部分。我们会显式地在环面上完成维约化: 先处理纯引力，再处理通常构成高维超引力物质部分的 p -形式。随后我们会在“隐藏对称性”一节中讨论，几何对称群如何被所谓的隐藏对称性扩充。

Geometric Symmetries

几何对称性

The dimensional reduction of supergravity can be performed most conveniently by using the vielbein formalism (see, e.g., [39]). We will first consider the reduction of pure gravity in an $(D+d)$ -dimensional spacetime on a d -dimensional torus T^d down to D dimensions. The coordinates of $(D+d)$ -dimensional spacetime are split according to

超引力的维约化最便于用标架形式化完成 (例如参见文献 [39])。我们首先考虑将纯引力从 $(D+d)$ 维时空约化到 d 维环面 T^d 上，最终得到 D 维理论。$(D+d)$ 维时空的坐标按如下方式拆分

$$x^M \to (x^\mu, y^m),\ \mu = 0,\dots,D-1,\ m = 1,\dots,d, \tag{41}$$

and similarly we split the flat Lorentz indices as

类似地，我们将平坦洛伦兹指标拆分为

$$A \to \left(a, \underline{a}\right),\ a = 0,\dots,D-1,\ \underline{a} = 1,\dots,d. \tag{42}$$

In toroidal dimensional reduction, all fields are taken to be independent of the coordinates y^m of the d -torus

在环形维约化中，我们认为所有场都不依赖于 d 维环面的坐标 y^m

$$\partial_m \Phi = 0. \tag{43}$$

One may think of a normal mode expansion of the fields and drop all modes other than the zero modes.

可以将场做正规模展开，仅保留零模，丢弃所有其他模。

The local Lorentz invariance in $(D+d)$ dimensions can be used to bring the vielbein into a triangular form

可以利用 $(D+d)$ 维的局域洛伦兹不变性将标架化为三角形式

$$E_M{}^A = \begin{pmatrix} E_\mu{}^a & E_\mu{}^{\underline{a}} \\ 0 & E_m{}^{\underline{a}} \end{pmatrix}, \tag{44}$$

which breaks the Lorentz group $\mathrm{SO}(1, D+d-1)$ down to $\mathrm{SO}(1, D-1) \times \mathrm{SO}(d)$. It turns out to be convenient to further parametrize (44) as

该操作将洛伦兹群 $\mathrm{SO}(1, D+d-1)$ 破缺为 $\mathrm{SO}(1, D-1) \times \mathrm{SO}(d)$ 。我们可以进一步将 (44) 参数化为如下便利形式

$$E_M{}^A = \begin{pmatrix} \mathrm{e}^{\gamma\phi} e_\mu{}^a & \mathrm{e}^{\phi/d} V_m{}^{\underline{a}} A_\mu{}^m \\ 0 & \mathrm{e}^{\phi/d} V_m{}^{\underline{a}} \end{pmatrix}, \tag{45}$$

with a matrix $V_m{}^{\underline{a}} \in \mathrm{SL}(d)$ of unit determinant, such that $\mathrm{e}^\phi = \det E_m{}^{\underline{a}}$. The constant

其中矩阵 $V_m{}^{\underline{a}} \in \mathrm{SL}(d)$ 的行列式为 1，且满足 $\mathrm{e}^\phi = \det E_m{}^{\underline{a}}$ 。常数

$$\gamma = -\frac{1}{D-2} \tag{46}$$

is chosen such that plugging (45) into the $(D+d)$ -dimensional Einstein-Hilbert Lagrangian, one finds

的选取保证将 (45) 代入 $(D+d)$ 维爱因斯坦-希尔伯特拉格朗日量后，可以得到

$$\mathcal{L}_{\mathrm{EH}}^{(D+d)} = |E|\, R_{(D+d)} \to |e|\, R_{(D)} + \ldots, \tag{47}$$

where $R_{(D)}$ is the Ricci scalar computed from the D -dimensional vielbein $e_\mu{}^a$, i.e., the reduced theory is directly obtained in the Einstein frame (without a dilaton prefactor). The ellipsis in (47) represents the matter couplings in the D -dimensional theory, i.e., the couplings of vector fields $A_\mu{}^m$ and scalar fields $\phi, V_m{}^{\underline{a}}$, from (45). Before working out the explicit form of these terms, it is instructive to analyze the symmetries of the lower-dimensional theory (47).

其中 $R_{(D)}$ 是由 D 维标架 $e_\mu{}^a$ 计算得到的里奇标量，也就是说约化理论直接在爱因斯坦框架下得到(无需伸缩子前置因子)。(47) 中的省略号代表 D 维理论中的物质耦合，即来自 (45) 的矢量场 $A_\mu{}^m$ 和标量场 $\phi, V_m{}^{\underline{a}}$ 的耦合。在推导出这些项的具体形式之前，分析低维理论 (47) 的对称性很有启发性。

The D -dimensional theory inherits a number of symmetries from its higher-dimensional ancestor. Namely, with the vielbein (45) transforming under infinitesimal diffeomorphisms as

D 维理论从高维原型继承了若干对称性。具体来说，标架 (45) 在无穷小微分同胚下的变换为

$$\delta_\xi E_M{}^A = \xi^N \partial_N E_M{}^A + E_N{}^A \partial_M \xi^N, \tag{48}$$

it is straightforward to see that such diffeomorphisms survive in the truncated theory (43) if the diffeomorphism parameter ξ^M itself satisfies (43). In particular, diffeomorphisms of the type $\xi^M = \{\xi^\mu(x), 0\}$ generate the D -dimensional diffeomorphisms on the fields $e_\mu{}^a, A_\mu{}^n, \phi$, and $V_m{}^{\underline{a}}$. On the other hand, under diffeomorphisms of the type $\xi^M = \{0, \xi^m(x)\}$, the fields $A_\mu{}^m$ transform as

不难发现，如果微分同胚参数 ξ^M 本身满足 (43)，这类微分同胚就能在截断理论 (43) 中保留下来。具体而言，$\xi^M = \{\xi^\mu(x), 0\}$ 形式的微分同胚会生成场 $e_\mu{}^a, A_\mu{}^n, \phi$ 和 $V_m{}^{\underline{a}}$ 上的 D 维微分同胚。另一方面，在 $\xi^M = \{0, \xi^m(x)\}$ 形式的微分同胚下，场 $A_\mu{}^m$ 的变换为

$$\delta A_\mu{}^m = \partial_\mu \xi^m(x), \tag{49}$$

whereas the graviton and the scalar fields are left inert. This shows that the resulting theory is an Abelian $\mathrm{U}(1)^d$ gauge theory with gauge fields $A_\mu{}^m$, while none of the matter is charged under the gauge group. Accordingly, the vector fields will couple with a Maxwell-type term in the reduced theory.

而引力子和标量场保持不变。这说明最终得到的理论是以 $A_\mu{}^m$ 为规范场的阿贝尔 $\mathrm{U}(1)^d$ 规范理论，且所有物质都不在规范群下带荷。因此，矢量场在约化理论中会以麦克斯韦型项耦合。

A different type of symmetry can be inferred from internal diffeomorphisms linear in the compactified coordinates y^m, i.e., of the form

我们可以从紧致坐标 y^m 上的线性内微分同胚得到另一类对称性，即形如

$$\xi^m(y) = \Lambda^m{}_n y^n. \tag{50}$$

Despite this dependence on the internal coordinates, the action (48) of such a diffeomorphism remains compatible with the truncation (43). Explicitly, this induces a global symmetry $\mathrm{SL}(d)$ acting on the D-dimensional matter fields parametrizing (45) as

尽管微分同胚依赖于内坐标，这类微分同胚的作用量 (48) 仍与截断 (43) 相容。具体而言，这会诱导出整体对称性 $\mathrm{SL}(d)$，作用在参数化 (45) 的 D 维物质场上，形式为

$$\delta V_m{}^{\underline{a}} = \Lambda^n{}_m V_n{}^{\underline{a}},\ \delta A_\mu{}^m = -\Lambda^m{}_n A_\mu{}^n, \tag{51}$$

where we have taken the matrix $\Lambda^m{}_n$ to be traceless $\Lambda^m{}_m \equiv 0$. The trace part of such transformation is slightly more subtle. With the parametrization (45), an internal diffeomorphism $\xi^m(y) \propto y^m$ also induces a nontrivial action on the D-dimensional vielbein. It has to be accompanied by the action of the trombone rescaling symmetry of the $(D+d)$-dimensional theory, c.f. (14), in order to yield a proper off-shell symmetry of the D-dimensional theory. The combination of these transformations induce the action

其中我们已经取矩阵 $\Lambda^m{}_n$ 为无迹的 $\Lambda^m{}_m \equiv 0$。这类变换的迹部分处理起来要更微妙。在参数化 (45) 下，内微分同胚 $\xi^m(y) \propto y^m$ 也会对 D 维标架产生非平凡作用。它必须伴随 $(D+d)$ 维理论的长号缩放对称性作用 (参见 (14))，才能得到 D 维理论的一个合适的离壳对称性。这些变换的组合会诱导出如下作用

$$\delta\phi = -\lambda,\ \delta A_\mu{}^m = \lambda\beta A_\mu{}^m,\ \beta = \frac{D+d-2}{d(D-2)}, \tag{52}$$

with constant λ, on the D-dimensional fields. Together, the transformations (51) and (52) generate a global $\mathrm{GL}(d)$ symmetry of the D-dimensional theory. We will refer to this group as the geometric symmetries of the theory, as they have their origin in the diffeomorphisms on the internal torus. Let us stress that the enhancement from $\mathrm{SL}(d)$ to $\mathrm{GL}(d)$ requires the higher-dimensional scaling symmetry (14) and is no longer realized in the presence of higher curvature corrections.

其中 λ 为常数，作用在 D 维场上。综上，变换 (51) 和 (52) 生成了 D 维理论的整体 $\mathrm{GL}(d)$ 对称性。我们将该群称为理论的几何对称性，因为它们起源于内环面上的微分同胚。需要强调，从 $\mathrm{SL}(d)$ 到 $\mathrm{GL}(d)$ 的对称性增强需要依赖高维缩放对称性 (14)，且在存在高阶曲率修正时无法成立

Finally, local Lorentz invariance is also a symmetry of the higher-dimensional theory. As mentioned above, the upper triangular form (44) of the vielbein breaks the original $\mathrm{SO}(1, D+d-1)$ down to $\mathrm{SO}(1, D-1)\times \mathrm{SO}(d)$, of which the first factor acts as D -dimensional Lorentz transformation on $e_\mu{}^a$ and the second factor acts as an additional local symmetry on $V_m{}^a$

最后，定域洛伦兹不变性也是高维理论的一个对称性。如上所述，标架的上三角形式 (44) 将原本的 $\mathrm{SO}(1, D+d-1)$ 破缺为 $\mathrm{SO}(1, D-1)\times \mathrm{SO}(d)$ ，其中第一个因子作为 D 维洛伦兹变换作用在 $e_\mu{}^a$ 上，第二个因子则对 $V_m{}^a$ 给出一个额外的定域对称性

$$\delta V_m{}^{\underline{a}} = V_m{}^{\underline{b}} \Lambda_{\underline{b}}{}^{\underline{a}}(x)\,,\ \Lambda(x) \in \mathfrak{so}(N)\,. \tag{53}$$

This shows that not all components of the matrix $V_m{}^a$ correspond to physical scalars as we make explicit in the following.

这说明并非矩阵 $V_m{}^a$ 的所有分量都对应物理标量，我们将在下文具体说明。

Reduction of Pure Gravity

纯引力的约化

For pure gravity, the fields of the D -dimensional theory are the various components of the higher-dimensional vielbein (45). The global and local symmetries of the D -dimensional theory, identified in the previous subsection, almost uniquely fix the form of the resulting two-derivative action. Explicitly, plugging the ansatz (43), (45), into the $(D+d)$ -dimensional Einstein-Hilbert Lagrangian, yields the completion of (47)

对于纯引力而言，D 维理论的场就是高维度标架 (45) 的各个分量。前一小节已经确定了 D 维理论的整体对称性和局域对称性，这些对称性几乎唯一确定了最终得到的二阶导数作用量的形式。具体来说，将假设式 (43)、(45) 代入 $(D+d)$ 维爱因斯坦-希尔伯特拉格朗日量，就得到了 (47) 的完备形式

$$\begin{aligned}\mathcal{L}_{\mathrm{EH}}^{(D+d)} = |E|\, R_{(D+d)} \to |e|\, R_{(D)} - |e| \operatorname{Tr}\left[P_\mu P^\mu\right] - \beta\, |e|\, \partial_\mu \phi \partial^\mu \phi \\ -\frac{1}{4}\, |e|\, \mathrm{e}^{2\beta\phi} M_{mn} F_{\mu\nu}{}^m F^{\mu\nu n} \\ + \text{total derivatives}, \end{aligned} \tag{54}$$

with β from (52). The vector fields appear with Abelian field strengths given by $F_{\mu\nu}{}^m = 2\partial_{[\mu}A_{\nu]}{}^m$ and a Maxwell term with the scalar dependent metric

其中 β 来自式 (52)。矢量场对应阿贝尔场强，由 $F_{\mu\nu}{}^m = 2\partial_{[\mu}A_{\nu]}{}^m$ 给出，并且带有一个依赖标量的度量的麦克斯韦项

$$M_{mn} = V_m{}^{\underline{a}} V_n{}^{\underline{b}} \delta_{ab}. \tag{55}$$

The scalar fields $V_m{}^{\underline{a}}$ describe a coset space sigma model

标量场 $V_m{}^{\underline{a}}$ 描述一个陪集空间 sigma 模型

$$\mathrm{G/K} = \mathrm{GL}(d)/\mathrm{SO}(d), \tag{56}$$

with the $\mathfrak{sl}(d)$ currents defined by

$\mathfrak{sl}(d)$ 流定义为

$$\delta^{\underline{ac}}(V^{-1})_{\underline{c}}{}^m \partial_\mu V_m{}^{\underline{b}} \equiv Q_\mu{}^{[\underline{ab}]} + P_\mu{}^{(\underline{ab})}, \tag{57}$$

and decomposed into their antisymmetric and symmetric parts. The target space $\mathrm{SL}(d)/\mathrm{SO}(d)$ is given by

并分解为其反对称部分和对称部分。目标空间 $\mathrm{SL}(d)/\mathrm{SO}(d)$ 由下式给出

$$-\mathrm{Tr}\left[P_\mu P^\mu\right] = \frac{1}{4}\partial_\mu M_{mn}\partial^\mu (M^{-1})^{mn}, \tag{58}$$

with the matrix M_{mn} from (55). Indeed, this kinetic term is invariant under the local symmetry transformations (53), showing that the matrix $V_m{}^a$ carries

其中矩阵 M_{mn} 来自式 (55)。事实上，该动能项在局域对称变换 (53) 下保持不变，这说明矩阵 $V_m{}^a$ 承载了

$$\dim(\mathrm{SL}(d)/\mathrm{SO}(d)) = \frac{1}{2}(d-1)(d+2), \tag{59}$$

physical scalar fields.

物理标量场。

Up to its relative coefficients, the Lagrangian (54) is the unique two-derivative Lagrangian for this field content, which is compatible with the global symmetries (51), (52), as well as with the gauge symmetries (49), (53), whose presence was deduced from the higher-dimensional diffeomorphism and Lorentz symmetries.

仅考虑相对系数的话，拉格朗日量 (54) 是该场内容唯一的二阶导数拉格朗日，它同时满足整体对称性 (51)、(52)，以及规范对称性 (49)、(53)，这些对称性是从高维微分同胚对称性和洛伦兹对称性推导得出的。

Reduction of p -Forms

p -形式的约化

The bosonic sector of higher-dimensional supergravities typically combines Einstein gravity with p -form matter couplings, such as the 3-form couplings (10) of 11D supergravity. Upon dimensional reduction on a torus T^d , this matter sector gives rise to additional fields, couplings, and symmetries in the lower-dimensional theory. Before performing the explicit toroidal reduction of the Lagrangian (10), it is instructive to first study the behavior of the additional fields w.r.t. the global $\mathrm{GL}(d)$ symmetry (51),(52).

高维超引力的玻色子 sector 通常结合了爱因斯坦引力与 p -形式物质耦合，例如 11 维超引力的 3-形式耦合 (10)。在环面 T^d 上完成维约化后，该物质 sector 会在低维理论中产生额外的场、耦合与对称性。在对拉格朗日量 (10) 进行具体的环面约化之前，先研究这些额外场相对于整体 $\mathrm{GL}(d)$ 对称性 (51)、(52) 的行为是具有启发性的。

Although we will mostly be interested in the reduction of a 3-form from 11 dimensions, the analysis is straightforward in general spacetime dimensions. Let us consider a p -form $C_{M_1\cdots M_p}$ in $(D+d)$ spacetime dimensions, such that its various components give rise to D -dimensional $p, p-1, \ldots, (p-d)$ -forms. The precise reduction ansatz corresponds to a split in the flat basis (42), i.e., the D -dimensional k -forms are built as

尽管我们主要关注 11 维中 3-形式的约化，但该分析在一般时空维度下都很直接。我们考虑 $(D+d)$ 维时空中的一个 p -形式 $C_{M_1\cdots M_p}$ ，它的各类分量会产生 D 维的 $p, p-1, \ldots, (p-d)$ -形式。精确的约化 ansatz 对应于平坦基 (42) 下的分解，即 D 维的 k -形式构造为

$$A_{\mu_1\cdots\mu_k m_{k+1}\cdots m_p} = P_{\mu_1}{}^{M_1}\cdots P_{\mu_k}{}^{M_k} C_{M_1\cdots M_k m_{k+1}\cdots m_p}, \tag{60}$$

with $P_\mu{}^M \equiv \{\delta_\mu{}^\nu, -A_\mu{}^m\}$. This ansatz (60) is such that the lower-dimensional fields remain invariant under the Kaluza-Klein gauge transformations (49). The transformation behavior of these fields under the global $\mathrm{SL}(d)$ symmetry from (51) can be computed from the action of diffeomorphisms (50) and follows from their index structure in the internal indices $m_1, m_2, \ldots$, e.g.,

其中满足 $P_\mu{}^M \equiv \{\delta_\mu{}^\nu, -A_\mu{}^m\}$。该 ansatz(60) 保证了低维场在卡鲁扎-克莱因规范变换 (49) 下保持不变。这些场在 (51) 给出的整体 $\mathrm{SL}(d)$ 对称性下的变换性质，可以从微分同胚 (50) 的作用计算得到，由其内指标 $m_1, m_2, \ldots$ 的指标结构直接导出，例如:

$$\delta A_{\mu mn} = \Lambda^k{}_m A_{\mu kn} + \Lambda^k{}_n A_{\mu mk}, \quad \text{etc. .} \tag{61}$$

Their charge under the GL(1) symmetry from (52) is slightly more tedious to determine, since it involves the $(D+d)$ -dimensional trombone symmetry as discussed above. One obtains (see, e.g., [40])

它们在 (52) 给出的 GL(1) 对称性下的荷确定起来稍显繁琐，因为正如前文讨论，这涉及到 $(D+d)$ 维的长号对称性。可以得到 (例如参见文献 [40]):

$$\delta_\lambda A_{\mu_1\cdots\mu_k m_{k+1}\cdots m_p} = -\lambda\,(p\gamma + (p-k)\,\beta)\,A_{\mu_1\cdots\mu_k m_{k+1}\cdots m_p}, \tag{62}$$

with γ and β from (46),(52). The higher-dimensional tensor gauge symmetries

其中 γ 和 β 来自 (46)、(52)。高维张量规范对称性

$$\delta C_{M_1\cdots M_p} = p\partial_{[M_1}\Lambda_{M_2\cdots M_p]}, \tag{63}$$

give rise to the lower-dimensional gauge symmetries of the k -forms. Due to the reduction ansatz (60), these symmetries in general mix forms of different degree with a nonlinear action.

给出了 k -形式的低维规范对称性。由于约化 ansatz(60)，这些对称性通常会混合不同次数的形式，作用是非线性的。

For a sufficiently large torus, i.e., for $d \geq p$, the reduction (60) adds $\begin{pmatrix} d \\ p \end{pmatrix}$ scalar fields $A_{m_1\ldots m_p}$ to the scalar sector of the D -dimensional theory. In analogy to (50), the higher-dimensional tensor gauge transformations linear in the compactified coordinates

对于足够大的环面，即当 $d \geq p$ 时，约化 (60) 会给 D 维理论的标量 sector 增加 $\begin{pmatrix} d \\ p \end{pmatrix}$ 个标量场 $A_{m_1\ldots m_p}$ 。与 (50) 类似，高维张量规范变换关于紧致坐标是线性的，

$$\Lambda_{m_1\ldots m_{p-1}}(y) = \xi_{m_1\ldots m_p} y^{m_p} \tag{64}$$

induce additional global shift symmetries

会诱导额外的整体平移对称性

$$\delta_\xi A_{m_1\ldots m_p} = \xi_{m_1\ldots m_p}, \tag{65}$$

on these scalar fields. These symmetries enhance the global $\mathfrak{gl}(d)$ from (51),(52), to a non-semisimple algebra of the type

作用在这些标量场上。这些对称性将 (51)、(52) 给出的整体 $\mathfrak{gl}(d)$ 增强为如下类型的非半单李代数

$$\mathfrak{g}_{\text{nss}} = \mathfrak{g}_0 \oplus \mathfrak{n}_+, \tag{66}$$

where $\mathfrak{g}_0$ combines the geometric $\mathfrak{gl}(d)$ with other potential global symmetries of the higher-dimensional theory, while the nilpotent $\mathfrak{n}_+$ combines all the shifts of type (65). For example, for reductions from 11D supergravity, $\mathfrak{g}_0 = \mathfrak{gl}(d)$, whereas for reductions from IIB supergravity, $\mathfrak{g}_0 = \mathfrak{gl}(d) \oplus \mathfrak{sl}(2)$. The algebra

(66) is graded w.r.t. the $\mathfrak{gl}(1) \subset \mathfrak{gl}(d)$ of (52), under which $\mathfrak{g}_0$ are the zero modes. Moreover, it follows from (62) that all generators of $\mathfrak{n}_+$ have positive charge under $\mathfrak{gl}(1)$. We emphasize once more that all the global symmetries (66) of the lower-dimensional theory have a direct geometrical origin by the higher-dimensional local diffeomorphism and tensor gauge symmetries.

其中 $\mathfrak{g}_0$ 将几何 $\mathfrak{gl}(d)$ 与高维理论的其他潜在整体对称性结合在一起，而幂零 $\mathfrak{n}_+$ 则包含了所有 (65) 型平移。例如，对 11 维超引力约化而言，$\mathfrak{g}_0 = \mathfrak{gl}(d)$, 而对 IIB 型超引力约化而言，$\mathfrak{g}_0 = \mathfrak{gl}(d) \oplus \mathfrak{sl}(2)$ 。代数 (66) 相对于 (52) 的 $\mathfrak{gl}(1) \subset \mathfrak{gl}(d)$ 是分次的，$\mathfrak{g}_0$ 是该分次下的零模。此外，由 (62) 可推得 $\mathfrak{n}_+$ 的所有生成元在 $\mathfrak{gl}(1)$ 下都带正电荷。我们再次强调，低维理论的所有整体对称性 (66) 都有直接的几何起源，来自高维的局部微分同胚和张量规范对称性。

Let us also note that part of the shift symmetries (65) may arise from the dual higher-dimensional p -forms. For example, we have noted in (16) the existence of the dual 6-form in 11D supergravity. Upon toroidal reduction on a sufficiently large torus T^d with $d \geq 6$, the associated gauge symmetries induce shift symmetries (65) on the scalar fields descending from the 6-form. This indicates that the full symmetry algebra (66) in general is only visible after taking into account all the fields together with their duals.

另外请注意，部分平移对称性 (65) 可能源自对偶高维 p -形式。例如，我们在 (16) 中已经提到 11 维超引力中存在对偶 6-形式。当在维度足够大的环面 T^d (满足 $d \geq 6$) 上做环面约化时，相关的规范对称性会诱导出源于 6-形式的标量场上的平移对称性 (65)。这说明一般而言，只有将所有场与其对偶场都考虑在内，才能看到完整的对称代数 (66)。

Furthermore, the symmetry induced by the gauge transformations (64) does not only act on scalar fields via the shift (65) but may in general also have a nontrivial action on some of the p -forms. Consider the gauge transformations (64)

此外，由规范变换 (64) 诱导出的对称性不仅通过平移 (65) 作用在标量场上，一般还可以对部分 p -形式产生非平凡作用。我们来看规范变换 (64)

$$\Lambda_{m_1 m_2}(y) = \xi_{m_1 m_2 m_3} y^{m_3}, \tag{67}$$

in 11D supergravity. While they induce the shifts (65) on the scalar fields $A_{m_1 m_2 m_3}$, they also have a nontrivial action on the p -forms descending from the dual 6-form according to its gauge transformation (19). For example, the D -dimensional 3-forms $C_{\mu\nu\rho m_1 m_2 m_3}$, transform as

它出现在 11 维超引力中。这些变换除了会诱导标量场 $A_{m_1 m_2 m_3}$ 上的平移 (65)，还会依照其规范变换 (19)，对源于对偶 6-形式的 p -形式产生非平凡作用。例如，D 维 3-形式 $C_{\mu\nu\rho m_1 m_2 m_3}$ 的变换规则为

$$\delta C_{\mu\nu\rho} m_1 m_2 m_3 = -\frac{3}{2} C_{\mu\nu\rho} \xi_{m_1 m_2 m_3}, \tag{68}$$

and similar for the lower-rank forms.

更低秩的形式也有类似的变换。

As an illustration for the reduction of p -forms, let us perform the explicit reduction of the 3-form Lagrangian (10) of 11D supergravity. The reduction ansatz (60) identifies the D -dimensional scalars,1-forms,2-forms, and 3-form, as

作为对 p -形式约化的说明，我们来对 11 维超引力中 3-形式的拉格朗日量 (10) 做显式约化。约化假设 (60) 给出了 D 维标量、1-形式、2-形式和 3-形式的对应关系:

$$A_{mnk} = C_{mnk}$$

$$A_{\mu mn} = C_{\mu mn} - A_{\mu}{}^{k} C_{kmn},$$

$$A_{\mu\upsilon m} = C_{\mu\upsilon m} - 2A_{[\mu}{}^{n} C_{\upsilon]mn} + A_{\mu}{}^{n} A_{\upsilon}{}^{k} C_{mnk},$$

$$A_{\mu\upsilon\rho} = C_{\mu\upsilon\rho} - 3A_{[\mu}{}^{m} C_{\upsilon\rho]m} + 3A_{[\mu}{}^{m} A_{\upsilon}{}^{n} C_{\rho]mn} - A_{\mu}{}^{m} A_{\upsilon}{}^{n} A_{\rho}{}^{k} C_{mnk},$$

(69)in terms of the components of C_{KLM} . The y -independent eleven-dimensional gauge transformations (63) translate into

它可以用 C_{KLM} 的分量表示。不依赖 y 的 11 维规范变换 (63) 可以转化为

$$\delta A_{\mu mn} = \partial_{\mu} \Lambda_{mn}$$

$$\delta A_{\mu\upsilon m} = 2\partial_{[\mu} \Lambda_{\upsilon]m} - F_{\mu\upsilon}{}^{n} \Lambda_{mn}, \tag{70}$$

$$\delta A_{\mu\nu\rho} = 3\partial_{[\mu} \Lambda_{\nu\rho]} - 3F_{[\mu\nu}{}^{m} \Lambda_{\rho]m},$$

where the lower-dimensional gauge parameters have been embedded into Λ_{MN} in analogy to (60). Likewise, the gauge invariant field strengths in D dimensions are defined via

其中低维规范参数仿照 (60) 嵌入到了 Λ_{MN} 中。类似地，D 维下规范不变场强的定义为

$$F_{\mu_1 \cdots \mu_k m_{k+1} \cdots m_4} = 4P_{\mu_1}{}^{M_1} \cdots P_{\mu_k}{}^{M_k} \partial_{[M_1} C_{\dots M_k m_{k+1} \cdots m_4]}, \tag{71}$$

and take the explicit form

其显式形式为

$$F_{\mu nkl} = \partial_{\mu} A_{nkl}$$

$$F_{\mu\upsilon mn} = 2\partial_{[\mu} A_{\upsilon]mn} + F_{\mu\upsilon}{}^{k} A_{kmn}, \tag{72}$$

$$F_{\mu\nu\rho m} = 3\partial_{[\mu} A_{\upsilon\rho]m} + 3F_{[\mu\upsilon}{}^{n} A_{\rho]mn},$$

$$F_{\mu\upsilon\rho\sigma} = 4\partial_{[\mu} A_{\upsilon\rho\sigma]} + 6F_{[\mu\upsilon}{}^{m} A_{\rho\sigma]m}.$$

They satisfy nonstandard nonlinear Bianchi identities

它们满足非标准的非线性比安基恒等式

$$3\partial_{[\mu} F_{\upsilon\rho]mn} = 3F_{[\mu\upsilon}{}^{k} F_{\rho]kmn},$$

$$4\partial_{[\mu} F_{\upsilon\rho\sigma]m} = 6F_{[\mu\upsilon}{}^{n} F_{\rho\sigma]mn}, \tag{73}$$

$$5\partial_{[\mu} F_{\nu\rho\sigma\lambda]} = 10F_{[\mu\nu}{}^{m} F_{\rho\sigma\lambda]m}.$$

Putting everything together, the kinetic term for the 3-form (10) reduces to

整合所有结果后，3-形式 (10) 的动能项约化为

$$\mathcal{L}_{\text{kin}} = -\frac{1}{48} |E| \, F^{KLMN} F_{KLMN}$$

$$= -\frac{1}{48} |e| \left(\mathrm{e}^{-6\gamma\phi} F^{\mu\upsilon\rho\sigma} F_{\mu\upsilon\rho\sigma} + 4\mathrm{e}^{-(6\gamma+2\beta)\phi} M^{mn} F^{\mu\upsilon\rho}{}_{m} F_{\mu\upsilon\rho n} \right.$$

$$+ 6\mathrm{e}^{-(6\gamma+4\beta)\phi} M^{mn} M^{kl} F^{\mu\upsilon}{}_{mk} F_{\mu\upsilon nl}$$

$$\left. + 4\mathrm{e}^{-(6\gamma+6\beta)\phi} M^{mn} M^{kl} M^{pq} \partial^{\mu} A_{mkp} \partial_{\mu} A_{nlq} \right),$$

(74)

with γ and β from (46) and (52) and the scalar dependent matrix M^{mn} denoting the inverse of (55). The reduced Lagrangian provides the kinetic terms for the lower-dimensional forms. It is straightforward to check that the dilaton powers precisely ensure invariance of the action under the GL(1) scaling symmetry (52), (62). Also the invariance under constant shifts (65) is manifest.

其中 γ 和 β 取自 (46) 和 (52)，依赖标量的矩阵 M^{mn} 是 (55) 的逆矩阵。约化后的拉格朗日量给出了低维形式的动能项。不难验证，伸缩子的幂次恰好保证了作用量在 GL(1) 标度对称性 (52)、(62) 下不变，同时在常数平移 (65) 下的不变性也是显然的。

Finally, reduction of the Chern-Simons term in (10) gives rise to a lengthy topological term in D dimensions. It is most compactly described by writing the original Chern-Simons term as the boundary contribution of some twelve-dimensional integral of

最后，(10) 中陈-西蒙斯项的约化会在 D 维中导出一个冗长的拓扑项。用最紧凑的方式描述它的方法是将原始陈-西蒙斯项改写为某个十二维积分的边界贡献，该积分的被积函数为

$$d\mathcal{L}_{\text{top}} = F^{(4)} \wedge F^{(4)} \wedge F^{(4)} \tag{75}$$

and to reduce the r.h.s. of this equation in terms of the different components (72).

随后将该方程的右侧按照不同分量 (72) 做约化。

The straightforward toroidal reduction of 11D supergravity thus gives a lower-dimensional theory with a global symmetry group of the type (66) with semisimple part $\mathfrak{g}_0 = \mathfrak{gl}(d)$. Before we discuss the further enhancement of the global symmetry group by hidden symmetries in the next section, let us briefly spell out the case $d = 2$, i.e., the reduction to $D = 9$ maximal supergravity.

因此，对 11 维超引力做直接环面约化，会得到一个整体对称群形如 (66)、半单部分为 $\mathfrak{g}_0 = \mathfrak{gl}(d)$ 的低维理论。在下一节我们讨论隐藏对称性对整体对称群的进一步扩充之前，我们先来简要说明 $d = 2$ 的情况，即约化得到 $D = 9$ 极大超引力的情况。

Maximal $D = 9$ Supergravity

最大 $D = 9$ 超引力

We first consider the case $d = 2$, i.e., the reduction of 11D supergravity on a two-torus T^2 . From (46) and (52), we find the values $\beta = \frac{9}{14}, \gamma = -\frac{1}{7}$. From the general structure given in the previous sections, we read off the resulting Lagrangian in $D = 9$ dimensions as the sum of (54),(74), and a nine-dimensional CS term as

我们首先考虑 $d = 2$ 的情形，即 11 维超引力在二维环面 T^2 上的约化。结合式 (46) 和 (52)，我们得到结果 $\beta = \frac{9}{14}, \gamma = -\frac{1}{7}$ 。根据前文给出的一般结构，我们推导出 $D = 9$ 维下的最终拉格朗日量，它由式 (54)、(74) 和一个九维陈-西蒙斯项相加得到，即

$$\begin{aligned}
\mathcal{L}_{\mathrm{EH}}^{(9)} &= |e|\, R_{(9)} - |e| \operatorname{Tr}\left[P_\mu P^\mu\right] - \frac{7}{2}\, |e|\, \partial_\mu \phi \partial^\mu \phi \\
&- \frac{1}{4}\, |e|\, \mathrm{e}^{3\phi} M_{mn} F_{\mu\upsilon}{}^m F^{\mu\upsilon n} - \frac{1}{8}\, |e|\, \mathrm{e}^{-4\phi} F^{\mu\upsilon} F_{\mu\upsilon} \\
&- \frac{1}{12}\, |e|\, \mathrm{e}^{-\phi} M^{mn} F^{\mu\upsilon\rho}{}_m F_{\mu\upsilon\rho n} - \frac{1}{48}\, |e|\, \mathrm{e}^{2\phi} F^{\mu\upsilon\rho\sigma} F_{\mu\upsilon\rho\sigma} \\
&+ \mathcal{L}_{\mathrm{top}}
\end{aligned} \tag{76}$$

where for convenience, we have rescaled the dilaton as $\phi \to \frac{7}{3}\phi$ and furthermore set

其中为方便起见，我们将 dilaton 重新标度为 $\phi \to \frac{7}{3}\phi$ ，并进一步设定

$$A_{\mu mn} = A_\mu \varepsilon_{mn},\ F_{\mu\nu mn} = F_{\mu\nu} \varepsilon_{mn}. \tag{77}$$

As discussed above, the global symmetry group of this theory is given by

如前文所述，该理论的整体对称群为

$$\mathrm{GL}(2) = \mathrm{SL}(2) \times \mathrm{GL}(1)\,, \tag{78}$$

and there is no further symmetry enhancement, as shift symmetries of the type (65) are absent.

并且不存在进一步的对称增强，因为式 (65) 类型的平移对称并不存在。

The remarkable property of the theory (76) is the fact that the very same theory is obtained by reducing ten-dimensional IIB supergravity on S^1 . This is consistent with the fact that only a single maximal supermultiplet exists in $D = 9$ dimensions. For the IIB reduction, the origin of the global symmetry (78) is the geometric GL(1) of the circle, together with the SL(2) symmetry of the IIB theory. The fields from (76) have a different higher-dimensional origin according to the scheme discussed in sections "Reduction of Pure Gravity" and "Reduction of p-Forms", specifically

理论 (76) 的一个显著性质是: 将十维 IIB 超引力在 S^1 上约化也会得到完全相同的理论。这与 $D = 9$ 维仅存在一个最大超多重态的结论一致。对于 IIB 超引力约化而言，整体对称 (78) 来源于圆的几何 GL(1)，加上 IIB 理论自身的 SL(2) 对称。根据“纯引力的约化”和“p-形式的约化”两节讨论的框架，式 (76) 中的各场来源于不同的高维源头，具体为

$$\text{11D : metric: } \left\{g_{\mu\nu}, A_\mu{}^m, \phi, M_{mn}\right\}\,,$$

$$\text{3-form: } \left\{A_\mu, A_{\mu\nu m}, A_{\mu\nu\rho}\right\}\,,$$

$$\text{IIB : metric: } \left\{g_{\mu\nu}, A_\mu, \phi\right\}\,, \text{scalars: } \left\{M_{mn}\right\}\,,$$

$$\text{2-form: } \left\{A_\mu{}^m, A_{\mu\nu m}\right\}, \;\; \text{4-form: } \left\{A_{\mu\nu\rho}\right\}\,. \tag{79}$$

The presence of Chern-Simons terms in (12) and (33) is indispensable for the equivalence of the two theories after toroidal reduction. It gives rise to nontrivial Bianchi identities for dual fields (18) akin to those appearing after dimensional reduction (73), thus allowing for the identification of fields of different higher-dimensional origin (79). A detailed discussion of $D = 9$ supergravity and its higher-dimensional embeddings is given in [41].

式 (12) 和 (33) 中陈-西蒙斯项的存在对保证环面约化后两个理论等价是必不可少的。它为对偶场 (18) 给出了非平凡比安基恒等式，这与维数约化 (73) 后得到的形式类似，因此我们可以将不同高维源头的场对应起来 (79)。文献 [41] 已经给出了 $D = 9$ 超引力及其高维嵌入的详细讨论。

Hidden Symmetries

隐藏对称性

We have seen in the previous section that toroidal reduction of higher-dimensional supergravity theories induces lower-dimensional supergravity theories with manifest global symmetries descending from higher-dimensional diffeomorphism and tensor gauge symmetries, spanning an algebra of the type (66). It is one of the most remarkable facts about these theories that on top of the geometric symmetries (66), the lower-dimensional supergravities possess further so-called hidden global symmetries, which only become apparent after toroidal reduction and proper redefinition of the fields [6, 16-18].

我们在前一节已经看到，高维超引力理论的环面约化会诱导出低维超引力理论，这些理论具有从高维微分同胚和张量规范对称性继承而来的明显整体对称性，张成式 (66) 类型的代数。关于这些理论最值得注意的一点是，除了几何对称性 (66) 之外，低维超引力还拥有进一步的所谓隐藏整体对称性，这些对称性只有在完成环面约化并对场进行恰当重定义后才会显现 [6, 16-18]。

As part of the general pattern, dubbed the ”silver rules of supergravity” [19], the full algebra of global symmetries of the lower-dimensional supergravity is given by the extension of (66) into a semi-simple algebra

作为被称为“超引力银规则”[19] 的一般规律的一部分，低维超引力整体对称性的完整代数由式 (66) 扩张为半单代数得到

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{n}_+. \tag{80}$$

The hidden symmetries combine into a nilpotent algebra $\mathfrak{n}_-$, which completes (66) into a semisimple algebra. W.r.t. $\mathfrak{gl}(d)$, the generators of $\mathfrak{n}_-$ transform in the representation dual to the generators of $\mathfrak{n}_+$. In particular, they carry negative charge under $\mathfrak{gl}(1) \subset \mathfrak{gl}(d)$. The fact that the dimension and structure of the algebra of hidden symmetries precisely fits the expansion (80) depends of course strongly on the field content and the couplings of the higher-dimensional supergravity theory. For generic couplings, no such symmetry enhancement would occur. Already a different pre-factor in front of the Chern-Simons term (10) of 11D supergravity would prevent the symmetry enhancement in all lower-dimensional theories. This is where supersymmetry comes to play its role, although here we only focus on the bosonic sectors of theory. For example, the coefficients in (10) allowing the symmetry enhancement (80) are precisely the coefficients that were fixed by supersymmetry of the 11D action.

隐藏对称性组合成幂零代数 $\mathfrak{n}_-$ ，它将式 (66) 完备化为一个半单代数。相对于 $\mathfrak{gl}(d)$ ，$\mathfrak{n}_-$ 的生成元按与 $\mathfrak{n}_+$ 生成元对偶的表示变换，具体而言，它们在 $\mathfrak{gl}(1) \subset \mathfrak{gl}(d)$ 下携带负电荷。隐藏对称代数的维数和结构恰好匹配展开式 (80)，这当然强烈依赖于高维超引力理论的场内容和相互作用。对于一般的相互作用，不会出现这种对称性增强。11 维超引力的陈-西蒙斯项 (10) 只要改变前置因子，就会破坏所有低维理论中的对称性增强。这就是超对称性发挥作用的地方，尽管本文我们仅关注该理论的玻色子 sector。举例来说，使得对称性增强 (80) 得以实现的式 (10) 中的系数，正是被 11 维作用量的超对称性固定下来的系数。

Table 1 Global symmetry groups G and their compact subgroups K in maximal supergravity in the various spacetime dimensions. For $D = 2$, the group $E_{9(9)}$ is the (centrally extended) affine extension of the group $E_{8(8)}$, and $K(E_9)$ denotes its maximal compact subgroup

表 1 不同时空维度最大超引力中的整体对称群 G 及其紧致子群 K。对于 $D = 2$ ，群 $E_{9(9)}$ 是群 $E_{8(8)}$ 的 (中心扩张) 仿射扩张，$K(E_9)$ 表示其极大紧致子群

D	G/K
9	GL(2)/SO(2)
8	$(\mathrm{SL}(2)\times\mathrm{SL}(3))\,/\,(\mathrm{SO}(2)\times\mathrm{SO}(3))$
7	SL(5)/SO(5)
6	$\mathrm{SO}(5,5)\,/\,(\mathrm{SO}(5)\times\mathrm{SO}(5))$
5	$\mathrm{E}_{6(6)}/\mathrm{USp}(8)$
4	$\mathrm{E}_{7(7)}/\mathrm{SU}(8)$
3	$\mathrm{E}_{8(8)}/\mathrm{SO}(16)$
2	$\mathrm{E}_{9(9)}/\mathrm{K}(\mathrm{E}_9)$

The global symmetry algebra (80) acts on all fields of the theory. While on the p -forms its action is necessarily linear as imposed by compatibility with gauge symmetry, the action of the hidden symmetries on the scalar fields is in general nonlinear. It is most elegantly described by the isometries of the coset space

整体对称代数 (80) 作用在该理论的所有场上。根据规范对称性相容性要求，它在 p -形式上的作用必然是线性的，而隐藏对称性对标量场的作用一般是非线性的。该作用最简洁的描述是陪集空间的等距同构

$$\mathrm{G}/\mathrm{K}, \tag{81}$$

with $\mathrm{G} = \mathrm{Lie}\ \mathfrak{g}$ and K its maximal compact subgroup. For maximal supergravity, the resulting global symmetry groups G build the $\mathrm{E}_{d(d)}$ series of non-compact exceptional Lie groups in the Dynkin classification, with Dynkin diagram given in Fig. 1 (The subscripts in parentheses in $\mathrm{E}_{d(d)}$ specify the particular real form of the exceptional groups. Specifically, it denotes the difference between non-compact and compact generators of the associated algebra. For maximal supergravity the global symmetry groups always appear in their split form, i.e., the maximally non-compact form of the group.). For small values of d , the series degenerates into the classical Lie groups

其中 $\mathrm{G} =$ 对应李代数 $\mathfrak{g}$ ，K 是它的极大紧致子群。对于最大超引力，得到的整体对称群 G 构成 Dynkin 分类中的非紧致例外李群 $\mathrm{E}_{d(d)}$ 序列，其 Dynkin 图如图 1 所示 ($\mathrm{E}_{d(d)}$ 中括号内的下标指定了例外群的特定实形式，具体而言，它表示关联代数中非紧致生成元与紧致生成元的数目差。对于最大超引力，整体对称群总是以分裂形式出现，即该群的极大非紧致形式)。对于小的 d 值，该序列退化为经典李群

$$\mathrm{E}_{5(5)} \simeq \mathrm{SO}(5,5)\,,\ \mathrm{E}_{4(4)} \simeq \mathrm{SL}(5)\,,\ \mathrm{E}_{3(3)} \simeq \mathrm{SL}(3)\times\mathrm{SL}(2)\,, \tag{82}$$

as can be extrapolated from the general Dynkin diagram of Fig. 1. The full set of coset spaces is listed in Table 1.

这可以从图 1 的一般 Dynkin 图外推得到，全套陪集空间已列于表 1 中。

Before going through the various cases, we first briefly review the structure of such coset spaces and their isometries.

在讨论具体情形之前，我们先简要回顾这类陪集空间及其等距同构的结构。

Coset Spaces

陪集空间

After toroidal reduction, the scalar fields of maximal supergravity theories are most conveniently described by a coset space sigma model. We have already encountered this structure in section "Reduction of Pure Gravity" in the reduction of pure gravity on a torus T^d with the scalars parametrizing the target space $\mathrm{GL}(d)/\mathrm{SL}(d)$ according to (58). Including the higher-dimensional p -forms, this space gets enhanced by the additional scalar fields into a larger coset space

经环面约化后，最大超引力理论的标量场最适合用陪集空间 σ 模型描述。我们已经在“纯引力的约化”章节遇到过该结构, 即纯引力在环面 T^d 上约化时, 标量根据 (58) 参数化目标空间 $\mathrm{GL}(d)/\mathrm{SL}(d)$ 。纳入高维 p 形式后，额外的标量场将该空间拓展为更大的陪集空间

$$\mathrm{GL}(d)/\mathrm{SL}(d) \hookrightarrow \mathrm{G}/\mathrm{K}\,. \tag{83}$$

Here, G is the Lie group associated to the algebra $\mathfrak{g}$ in (80), and K denotes its maximal compact subgroup. The coset space is described by a representative (or vielbein) $\mathcal{V} \in \mathrm{G}$ with local gauge invariance

此处，G 是与 (80) 中代数 $\mathfrak{g}$ 对应的李群，K 表示其极大紧子群。陪集空间由具有局域规范不变性的代表元 (或标架) $\mathcal{V} \in \mathrm{G}$ 描述

$$\delta\mathcal{V} = \mathcal{V}k(x)\,,\ k(x) \in \mathfrak{k}, \tag{84}$$

with $\mathrm{K} = \mathrm{Lie}\ \mathfrak{k}$. In analogy to (57),(58), the Lagrangian for the coset space sigma model is built by decomposing the left invariant scalar currents as

其中满足 $\mathrm{K} =$ 为李 $\mathfrak{k}$ 。类比 (57) 和 (58)，陪集空间 σ 模型的拉格朗日量通过分解左不变标量流构造如下:

$$\mathcal{V}^{-1}\partial_\mu\mathcal{V} = Q_\mu + P_\mu, \tag{85}$$

with $Q_\mu \in \mathfrak{k}$ and $P_\mu \in \mathfrak{p}$, according to the orthogonal decomposition

其中 $Q_\mu \in \mathfrak{k}$ 和 $P_\mu \in \mathfrak{p}$ ，满足正交分解

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}. \tag{86}$$

The Lagrangian

拉格朗日量

$$\mathcal{L}_{\text{coset}} = -\frac{1}{2}|e|\,\mathrm{Tr}\left(P_\mu P^\mu\right), \tag{87}$$

then is invariant under the gauge transformations (84), under which

在规范变换 (84) 下保持不变，变换中

$$\delta P_\mu = \left[P_\mu, k(x)\right]. \tag{88}$$

The gauge symmetry (84) can be fixed by imposing a sufficient number of conditions on the vielbein $\mathcal{V}$, such that it is uniquely parametrized by

规范对称性 (84) 可通过对标架 $\mathcal{V}$ 施加足够多的条件来固定，使得标架由

$$n = \dim \mathrm{G} - \dim \mathrm{K}, \tag{89}$$

scalar fields, corresponding to a choice of coordinates on the target space (83). The Lagrangian (87) remains invariant under the global symmetry

标量场唯一参数化，对应目标空间 (83) 上的一种坐标选择。拉格朗日量 (87) 在整体对称性下仍保持不变

$$\delta_g \mathcal{V} = g\mathcal{V} - \mathcal{V} k_g,\ g \in \mathfrak{g},\ k_g \in \mathfrak{k}, \tag{90}$$

combining left multiplication on $\mathcal{V}$ with a compensating gauge transformation (84) in order to restore the fixed gauge. The action (90) describes the infinitesimal action of the isometry group G on the n coordinates of the target space (83). In particular, it encodes the action of the global symmetry group G on the fermionic sector of the theory. Before gauge fixing the local symmetry (84), the fermions of the theory appear as singlets under G but transform under local K transformations (84). Derivatives are covariantized with the composite connection Q_μ from (85). After gauge fixing, the fermions inherit a nontrivial action of the global symmetry group G by means of the compensating $\mathfrak{k}$ -transformation $k_g \in \mathfrak{k}$ of (90).

该整体对称性将 $\mathcal{V}$ 上的左乘与补偿规范变换 (84) 结合，以恢复固定规范。变换 (90) 描述了等距群 G 对目标空间 (83) 的 n 坐标的无穷小作用。特别地，它编码了整体对称群 G 在理论费米子部分的作用。在固定局域对称性 (84) 的规范之前，理论的费米子是 G 下的单态，但会按照局域 K 变换 (84) 变换，导数由 (85) 的复合联络 Q_μ 协变化。规范固定后，通过 (90) 中补偿的 $\mathfrak{k}$ 变换 $k_g \in \mathfrak{k}$ ，费米子获得了整体对称群 G 的非平凡作用。

In the context of toroidal reduction of supergravity, a natural gauge fixing for the vielbein $\mathcal{V}$ is the triangular gauge, in which this matrix is put to the form

在超引力环面约化的背景下，标架 $\mathcal{V}$ 一种自然的规范固定是三角规范，该规范下矩阵被化为如下形式

$$\mathcal{V} = \exp\left(\phi^a N_a\right) V_{\mathrm{G}_0},\ V_{\mathrm{G}_0} \in \mathrm{G}_0 = \mathrm{Lie}\, \mathfrak{g}_0, \tag{91}$$

where the right factor V_{G_0} lives in the Lie group associated with the algebra of zero charge generators $\mathfrak{g}_0$ in (80) and the N_a denote the generators of the nilpotent algebra $\mathfrak{n}_+$. The matrix V_{G_0} is built from the internal part of the higher-dimensional vielbein ${V_m}^{\underline{a}}$ introduced in (45) (together with the other scalars of the higher-dimensional theory), while the scalars ϕ^a describe the scalars descending from the higher-dimensional p -forms.

其中右因子 V_{G_0} 属于 (80) 中零荷生成元 $\mathfrak{g}_0$ 对应代数的李群，N_a 表示幂零代数 $\mathfrak{n}_+$ 的生成元。矩阵 V_{G_0} 由 (45) 引入的高维标架 ${V_m}^{\underline{a}}$ 的内部分 (结合高维理论的其他标量) 构造而来，标量 ϕ^a 描述来自高维 p 形式的标量。

As a result, the action (90) of $\mathfrak{g}_0$ induces a linear action on the scalar fields ϕ^a , in accordance with the representation of the associated generators N_a . The action (90) of $\mathfrak{n}_+$ does not require a compensating gauge transformation, $k_{\mathfrak{n}_+} = 0$, and induces shift symmetries on the scalars ϕ^a , generating the transformations (65). In contrast, the action (90) of the hidden symmetries $\mathfrak{n}_-$ induces a compensating gauge transformation and thereby a nonlinear action on the scalar fields.

因此，$\mathfrak{g}_0$ 的作用 (90) 对标量场 ϕ^a 诱导出线性作用，符合对应生成元 N_a 的表示。$\mathfrak{n}_+$ 的作用 (90) 不需要补偿规范变换 $k_{\mathfrak{n}_+} = 0$ ， 会对标量 ϕ^a 诱导出平移对称性，生成变换 (65)。相反，隐藏对称性 $\mathfrak{n}_-$ 的作用 (90) 会诱导出补偿规范变换，从而对标量场产生非线性作用。

As an illustration, let us evaluate the transformation (90) for the coset space SL (2) /SO (2) , which appears in the matter sector of various supergravity theories. With the $\mathfrak{sl}(2)$ algebra generators given by

举例来说，我们来计算陪集空间 SL (2) /SO (2) 的变换 (90)，该陪集空间出现在多种超引力理论的物质 sector 中。已知 $\mathfrak{sl}(2)$ 代数生成元为

$$\mathbf{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \mathbf{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \mathbf{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{92}$$

the decomposition (80) of the algebra corresponds to

该代数的分解 (80) 对应于

$$\mathfrak{sl}(2) = \mathfrak{n}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{n}_+ = \langle \mathbf{f} \rangle \oplus \langle \mathbf{h} \rangle \oplus \langle \mathbf{e} \rangle. \tag{93}$$

Accordingly, the matrix $\mathcal{V}$ in triangular gauge (91) can be parametrized as

相应地，三角规范 (91) 中的矩阵 $\mathcal{V}$ 可以参数化为

$$\mathcal{V} = \exp(C\mathbf{e}) \exp\left(\frac{1}{2}\phi \mathbf{h}\right) = \begin{pmatrix} \mathrm{e}^{\phi/2} & \mathrm{e}^{-\phi/2} C \\ 0 & \mathrm{e}^{-\phi/2} \end{pmatrix}. \tag{94}$$

The action (90) then induces the transformation

作用量 (90) 随后诱导出变换

$$\delta_{\mathbf{h}} \phi = 2, \ \delta_{\mathbf{h}} C = 2C, \ \delta_{\mathbf{e}} C = 1, \ \delta_{\mathbf{f}} \phi = -2C, \ \delta_{\mathbf{f}} C = \mathrm{e}^{2\phi} - C^2, \tag{95}$$

on the scalars ϕ, C . This shows how the algebra $\mathfrak{g}_0 = \langle \mathbf{h} \rangle$ acts as a scaling symmetry on the fields, whereas $\mathfrak{n}_+ = \langle \mathbf{e} \rangle$ acts as a shift symmetry on C . The hidden symmetries in this example are generated by $\mathfrak{n}_- = \langle \mathbf{f} \rangle$, inducing a nonlinear action on the scalar fields. Let us also note that the Lagrangian (87) for this example is given by

作用于标量场 ϕ, C 。这表明代数 $\mathfrak{g}_0 = \langle \mathbf{h} \rangle$ 如何对标量场发挥标度对称性的作用，而 $\mathfrak{n}_+ = \langle \mathbf{e} \rangle$ 则对 C 发挥平移对称性的作用。本示例中的隐藏对称性由 $\mathfrak{n}_- = \langle \mathbf{f} \rangle$ 生成，它诱导出对标量场的非线性作用。另请注意，本示例的拉格朗日量 (87) 为

$$\mathcal{L}_{\text{coset}} = -\frac{1}{4}\,|e|\,\partial_\mu \phi \partial^\mu \phi - \frac{1}{4}\,|e|\,\mathrm{e}^{-2\phi} \partial_\mu C \partial^\mu C, \tag{96}$$

which is invariant under the transformations (95).

它在变换 (95) 下保持不变。

This is the coset space that already appears in the ten-dimensional IIB super-gravity (35) (with $C_0 = C, \phi \to -\phi$). In lower-dimensional supergravities, this coset space shows up, for example, in the reduction of $D = 5$ minimal supergravity on a circle S^1 . The bosonic field content of the $D = 5$ theory comprises the metric and a single vector field. Its S^1 reduction follows the scheme described in sections "Reduction of Pure Gravity" and "Reduction of p -Forms", with $\gamma = -\frac{1}{2}, \beta = \frac{3}{2}$. In $D = 4$ dimensions, it gives rise to gravity coupled to two vectors and two scalar fields. According to the symmetry enhancement (80), the two scalars build an SL(2)/SO(2) sigma model, given by (96). The geometric symmetries (66) in this example contain the

这就是已经出现在十维 IIB 超引力 (35) 中的陪集空间 (带有 $C_0 = C, \phi \to -\phi$)。在低维超引力中，该陪集空间例如出现在 $D = 5$ 极小超引力沿圆周 S^1 的约化中。$D = 5$ 理论的玻色场内容包含度规和单个矢量场。它的 S^1 约化遵循“纯引力的约化”和“p -形式的约化”两节中描述的框架，带有 $\gamma = -\frac{1}{2}, \beta = \frac{3}{2}$ 。在 $D = 4$ 维中，该约化得到引力与两个矢量场和两个标量场耦合。根据对称性增强 (80)，两个标量场构成了 SL(2)/SO(2) sigma 模型，由式 (96) 给出。本示例中的几何对称性 (66) 包含

$$\mathfrak{gl}(1) = \mathfrak{g}_0 = \langle \mathbf{h} \rangle, \tag{97}$$

whose action (52),(62) on the scalar fields is reproduced by $-\frac{1}{2}\delta_{\mathbf{h}}$ in (95). The same formulas show that the two vector fields arising in this reduction carry charges $\left(+\frac{3}{2}\right)$ and $\left(+\frac{1}{2}\right)$, respectively, under $\mathfrak{gl}(1)$. Together with their dual vectors (defined via the duality equation $\widetilde{F}_{\mu\nu} = \frac{1}{2}\,|e|\,\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ in $D = 4$ dimensions) , they fill the spin- $\frac{3}{2}$ representation of SL(2). This illustrates the fact the realization of the full symmetry group in general involves the original and the dual fields of the theory.

其对标量场的作用 (52)、(62) 由 (95) 中的 $-\frac{1}{2}\delta_{\mathbf{h}}$ 重现。相同的公式表明，该约化得到的两个矢量场分别在 $\mathfrak{gl}(1)$ 下携带电荷 $\left(+\frac{3}{2}\right)$ 和 $\left(+\frac{1}{2}\right)$ 。结合它们的对偶矢量 (通过对偶方程 $\widetilde{F}_{\mu\nu} = \frac{1}{2}\,|e|\,\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ 在 $D = 4$ 维) 中定义)，它们填满了 SL(2) 的自旋- $\frac{3}{2}$ 表示。这说明一个事实: 一般而言，完整对称群的实现会同时涉及理论的原场和对偶场。

In a similar way, the coset space SL(2)/SO(2) appears as one of the factors of the scalar target space in $D = 8$ maximal supergravity, c.f. Table 1, as we shall discuss in more detail in section " $D = 8, 7, 6, 5$ Maximal Supergravities".

类似地，陪集空间 SL(2)/SO(2) 是 $D=8$ maximal 超引力中标量目标空间的因子之一，参见表 1，我们将在“$D=8,7,6,5$ 极大超引力”一节中进行更详细的讨论。

Yet another example featuring the same global symmetry group SL(2) comes from the S^1 reduction of four-dimensional Einstein gravity. In this case, the geometric scaling symmetry is still (97), but the shift symmetry $\delta_{\mathbf{e}}$ acts on the scalar that is obtained by dualizing the three-dimensional Kaluza-Klein vector A_μ . The nonlinear action of the hidden symmetry $\delta_{\mathbf{f}}$ in this example was originally discovered in [42], and this SL(2) group goes under the name of the Ehlers group. It is instructive to spell out some details of this first example of hidden symmetries. The S^1 reduction of $D=4$ gravity is described by an ansatz (45) with $\gamma=-1$, $\beta=2$. According to (54), it results in a three-dimensional Lagrangian (W.r.t. the conventions of (45),(54), we have rescaled $\phi\to\phi/2$.)

另一个具有相同整体对称群 SL(2) 的例子，来自四维爱因斯坦引力的 S^1 约化。在该情形下，几何标度对称性仍为 (97)，但平移对称性 $\delta_{\mathbf{e}}$ 作用在对偶化三维卡鲁扎-克莱因矢量 A_μ 得到的标量场上。该例子中隐藏对称性 $\delta_{\mathbf{f}}$ 的非线性作用最初发现于文献 [42]，这个 SL(2) 群被命名为埃勒斯群。详细说明这个第一个隐藏对称性例子会很有启发性。$D=4$ 引力的 S^1 约化由 ansatz (45) 描述，其中 $\gamma=-1$ 、$\beta=2$ 。根据 (54)，它得到三维拉格朗日量 (参照 (45)、(54) 的约定，我们对 $\phi\to\phi/2$ 做了重新标度。)

$$\mathcal{L}^{(3)}=|e|\,R_{(3)}-\frac{1}{2}\,|e|\,\partial_\mu\phi\partial^\mu\phi-\frac{1}{4}\,|e|\,\mathrm{e}^{2\phi}F_{\mu\nu}F^{\mu\nu},\tag{98}$$

which exhibits the GL(1) symmetry (52), scaling the scalar and the vector field. In three dimensions, vector fields are dual to scalar fields. This is most conveniently implemented in the Lagrangian (98) by treating $F_{\mu\nu}$ as a fundamental field (instead of the gauge potential A_μ) and implementing its Bianchi identity by means of a Lagrange multiplier C as

它展现出 GL(1) 对称性 (52)，对标量场和矢量场做标度变换。在三维中，矢量场对偶于标量场。这一点可以在拉格朗日量 (98) 中很方便地实现: 将 $F_{\mu\nu}$ 视为基本场 (而非规范势 A_μ)，再通过拉格朗日乘子 C 引入其比安基恒等式，得到

$$\mathcal{L}^{(3)}_{\text{parent}}=|e|\,R_{(3)}-\frac{1}{2}\,|e|\,\partial_\mu\phi\partial^\mu\phi-\frac{1}{4}\,|e|\,\mathrm{e}^{2\phi}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2}\varepsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho}C.\tag{99}$$

The resulting field equations for $F_{\mu\nu}$ are algebraic and can be used to eliminate this field, leading to the dual Lagrangian

最终得到的 $F_{\mu\nu}$ 场方程是代数型的，可用于消去该场，从而得到对偶拉格朗日量

$$\mathcal{L}^{(3)}_{\text{dual}}=|e|\,R_{(3)}-\frac{1}{2}\,|e|\,\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}\,|e|\,\mathrm{e}^{-2\phi}\partial_\mu C\partial^\mu C,\tag{100}$$

in terms of two scalar fields. This is precisely the coset space sigma model (96). After dualizing the three-dimensional vector A_μ into a scalar field, the symmetry of the Lagrangian is thus enhanced to SL(2). Uplifting its action (95) back to $D=4$ dimensions then induces a ”hidden” symmetry of general relativity acting on solutions with a U(1) isometry.

用两个标量场表示。这恰好就是陪集空间 sigma 模型 (96)。将三维矢量 A_μ 对偶化为标量场后，拉格朗日量的对称性因此提升为 SL(2)。再将其作用 (95) uplifting 回 $D=4$ 维，就会得到广义相对论的一个“隐藏”对称性，作用在具有 U(1) 等距的解上。

$D = 8, 7, 6, 5$ Maximal Supergravities

$D = 8, 7, 6, 5$ 最大超引力

We have seen in section ”Maximal $D = 9$ Supergravity” that the reduction of 11D supergravity on a two-torus T^2 leads to maximal $D = 9$ supergravity whose global symmetries (78) do not include any hidden symmetries but are limited to the geometric symmetries $\mathfrak{g}_0$ in (66). Hidden symmetries appear upon reduction on larger tori.

我们已经在“最大 $D=9$ 超引力”一节中看到，11 维超引力在二维环面 T^2 上约化得到最大 $D=9$ 超引力，其整体对称性 (78) 不包含任何隐对称性，仅局限于 (66) 中的几何对称性 $\mathfrak{g}_0$ 。隐对称性会在更大环面上约化时出现。

Let us start with $D = 8$ maximal supergravity, obtained by reduction of 11D supergravity on T^3 [43]. The necessity of a symmetry enhancement in this theory can already be deduced from simple group theory considerations. As discussed above, toroidal reduction of 11D supergravity on a three-torus T^3 yields a theory with manifest geometric symmetries (66)

我们从 11 维超引力在 T^3 上约化得到的 $D=8$ 最大超引力开始讨论 [43]。该理论需要对称性增强这一点可以从简单的群论考虑推导得出。如上文所述，11 维超引力在三维环面 T^3 上的环面约化得到的理论具有明显几何对称性 (66)

$$\mathfrak{gl}(3) \oplus 1_+ \,, \tag{101}$$

where 1_+ denotes the one-dimensional algebra $\mathfrak{n}_+$ generated by shifts (65) on the scalar descending from the 11D 3-form. On the other hand, it follows from the discussion of section ”Maximal $D = 9$ Supergravity” that the same $D = 8$ supergravity is obtained by reducing IIB supergravity on a two-torus T^2 . In this case, the higher-dimensional origin implies a global symmetry group (66)

其中 1_+ 表示由 11 维 3 形式派生的标量上的平移 (65) 生成的一维代数 $\mathfrak{n}_+$ 。另一方面，从“最大 $D=9$ 超引力”一节的讨论可知，同一 $D=8$ 超引力也可以通过 IIB 超引力在二维环面 T^2 上约化得到。在这种情况下，更高维的起源暗示整体对称群为 (66)

$$(\mathfrak{gl}(2) \oplus \mathfrak{sl}(2)) \oplus 2_+, \tag{102}$$

where $\mathfrak{sl}(2)$ is the symmetry of the IIB theory and 2_+ denotes the two-dimensional algebra $\mathfrak{n}_+$ generated by shifts (65) on the scalars descending from the doublet of IIB 2-forms. Thus, the full global symmetry algebra of $D = 8$ supergravity must unite both, (101) and (102). This is realized by the group

其中 $\mathfrak{sl}(2)$ 是 IIB 理论的对称性，2_+ 表示由 IIB 二重 2 形式派生的标量上的平移 (65) 生成的二维代数 $\mathfrak{n}_+$ 。因此，$D = 8$ 超引力的完整整体对称代数必须同时包含 (101) 和 (102)，这由以下群实现

$$\mathrm{E}_{3(3)} \simeq \mathrm{SL}(3) \times \mathrm{SL}(2) \tag{103}$$

whose algebra admits the decompositions

其代数满足分解

$$\mathfrak{e}_{3(3)} = \mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \overset{11\mathrm{D}}{\to} 1_- \oplus \mathfrak{gl}(3) \oplus 1_+, \tag{104}$$

$$\mathfrak{e}_{3(3)} = \mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \overset{\mathrm{IIB}}{\to} 2_- \oplus (\mathfrak{gl}(2) \oplus \mathfrak{sl}(2)) \oplus 2_+,$$

embedding both (101) and (102), in accordance with (80). With the labelling (92) of $\mathfrak{sl}(2)$ generators, the geometric symmetries (101) from 11D supergravity are identified as

按照 (80) 的要求同时嵌入了 (101) 和 (102)。利用 $\mathfrak{sl}(2)$ 生成元的标记 (92)，11 维超引力得到的几何对称性 (101) 可识别为

$$\langle \mathbf{h} \rangle = \mathfrak{gl}(1) \subset \mathfrak{gl}(3), \ \langle \mathbf{e} \rangle = 1_+. \tag{105}$$

The scalars of $D = 8$ maximal supergravity descend from the internal block of the 11D metric together with the single scalar C descending from the 11D 3-form. The symmetry enhancement (103) corresponds to the enhancement of the coset space (56) to

$D = 8$ 最大超引力的标量由 11 维度规的内禀块，加上 11 维 3 形式派生的单个标量 C 共同得到。对称性增强 (103) 对应陪集空间 (56) 增强为

$$\mathrm{GL}(3)/\mathrm{SO}(3) \hookrightarrow (\mathrm{SL}(3)/\mathrm{SO}(3)) \times (\mathrm{SL}(2)/\mathrm{SO}(2)). \tag{106}$$

In particular, the determinant of the internal metric combines with the scalar C into an SL(2)/SO(2) coset space sigma model, similar to the example of minimal $D = 5$ supergravity discussed after (96) above. As another nontrivial consequence of the symmetry enhancement, the vector fields of $D = 8$ supergravity

具体而言，内禀度规的行列式与标量 C 结合为 SL(2)/SO(2) 陪集空间 σ 模型，类似于上文 (96) 后讨论的最小 $D = 5$ 超引力的例子。对称性增强的另一非平凡结果是，$D = 8$ 超引力的矢量场

$$\left\{A_\mu{}^m, A_{\mu mn}\right\}, \ m, n = 1, 2, 3, \tag{107}$$

descending from the 11D metric and 3-form, respectively, combine into an SL(2) doublet, i.e., span the $(3, 2)$ representation of $\mathrm{SL}(3)\times\mathrm{SL}(2)$. Indeed, one may check with (52), (62), and the identification (105) that these fields have opposite charges $\pm\frac{1}{2}$ under $\mathbf{h} \in \mathfrak{sl}(2)$. Moreover, under $\mathbf{e}$, acting as shift symmetry (65) on the scalar C , the vectors $A_{\mu mn}$ transform into $A_\mu{}^m$, following the higher-dimensional embedding (69). The $D = 8$,2-forms $A_{\mu\nu m}$ have zero charge under $\mathbf{h} \in \mathfrak{sl}(2)$ and remain SL(2) singlets. For the $D = 8, 3$ -form, the symmetry enhancement can only be made visible upon including the dual fields. To this end, consider the 3-form $A_{\mu\upsilon\rho}$ descending from the 11D three form, together with its dual

分别由 11 维度规和 3 形式派生，结合为 SL(2) 二重态，即张成 $\mathrm{SL}(3)\times\mathrm{SL}(2)$ 的 $(3,2)$ 表示。实际上利用 (52)、(62) 和等价关系 (105) 可以验证，这些场在 $\mathbf{h}\in\mathfrak{sl}(2)$ 下带相反电荷 $\pm\frac{1}{2}$ 。此外，在以平移对称 (65) 作用于标量 C 的 $\mathbf{e}$ 下，矢量 $A_{\mu mn}$ 变换为 $A_\mu{}^m$ ，符合高维嵌入 (69)。$D=8$ 即 2 形式 $A_{\mu\upsilon m}$ 在 $\mathbf{h}\in\mathfrak{sl}(2)$ 下电荷为零，仍是 SL(2) 单态。对于 $D=8,3$ 形式，只有引入对偶场才能看出对称性增强。为此，我们考虑 11 维三形式派生的三形式 $A_{\mu\upsilon\rho}$ ，连同其对偶

$$\widetilde{A}_{\mu\upsilon\rho}=\frac{1}{6}A_{\mu\upsilon\rho kmn}\varepsilon^{kmn}, \tag{108}$$

descending from the 11D 6-form (Equivalently, in $D=8$ dimensions, the two 3-forms are related by a first-order duality equation of the form $\widetilde{F}_{\mu_1\ldots\mu_4}=\frac{1}{24}|e|\,\varepsilon_{\mu_1\ldots\mu_8}F^{\mu_5\cdots\mu_8}$, obtained by dimensional reduction of (16).). Equation (62) shows that they have opposite charges $\pm\frac{1}{2}$ under $\mathbf{h}$, while (68) shows how they are mapped into each other under the action of $\mathbf{e}$. This shows that the 3-form $A_{\mu\upsilon\rho}$ together with its dual forms a doublet under the SL(2). Once more, this illustrates that the realization of the full enhanced symmetry group in general involves the original and the dual fields of the theory.

源自 11 维 6-form(等价地，在 $D=8$ 维中，两个 3-form 满足形如 $\widetilde{F}_{\mu_1\ldots\mu_4}=\frac{1}{24}|e|\,\varepsilon_{\mu_1\ldots\mu_8}F^{\mu_5\cdots\mu_8}$ 的一阶对偶方程，由式 (16) 维度约化得到)。式 (62) 表明它们在 $\mathbf{h}$ 下带有相反电荷 $\pm\frac{1}{2}$ ，而式 (68) 给出了它们在 $\mathbf{e}$ 作用下的相互映射关系。这说明 3-form $A_{\mu\upsilon\rho}$ 与其对偶共同构成 SL(2) 下的二重态。这再一次说明，完整增强对称群的实现通常需要同时包含理论的原始场与对偶场。

For the lower-dimensional maximal supergravities, the symmetry enhancement proceeds in an analogous way. For $D=7$ maximal supergravity [44], the geometric symmetries enhance to an $\mathrm{E}_{4(4)}=\mathrm{SL}(5)$ global symmetry group, with the decompositions (80) corresponding to (For the $\mathrm{SL}(d)$ groups, we use the notation R' to denote the dual representation to R .)

对于更低维的最大超引力，对称增强以类似方式发生。对于 $D=7$ 维最大超引力 [44]，几何对称性增强为 $\mathrm{E}_{4(4)}=\mathrm{SL}(5)$ 整体对称群，分解式 (80) 对应于 (对于 $\mathrm{SL}(d)$ 群，我们沿用记号 R' 表示 R 的对偶表示)

$$\mathfrak{e}_{4(4)}=\mathfrak{sl}(5)\xrightarrow{\text{11D}}4'_-\oplus\mathfrak{gl}(4)\oplus 4_+, \tag{109}$$

$$\mathfrak{e}_{4(4)}=\mathfrak{sl}(5)\xrightarrow{\text{IIB}}(3',2)_-\oplus(\mathfrak{gl}(3)\oplus\mathfrak{sl}(2))\oplus(3,2)_+,$$

respectively, depending on the higher-dimensional origin.

依高维起源不同分别对应。

For $D=6$ maximal supergravity [45], the geometric symmetries enhance to an $\mathrm{E}_{5(5)}=\mathrm{SO}(5,5)$ global symmetry group, with the decompositions (80) corresponding to

对于 $D=6$ 维最大超引力 [45]，几何对称性增强为 $\mathrm{E}_{5(5)}=\mathrm{SO}(5,5)$ 整体对称群，分解式 (80) 对应于

$$\mathfrak{e}_{5(5)} = \mathfrak{so}(5,5) \xrightarrow{\text{11D}} 10'_{-} \oplus \mathfrak{gl}(5) \oplus 10_{+}, \tag{110}$$

$$\mathfrak{e}_{5(5)} = \mathfrak{so}(5,5) \xrightarrow{\text{IIB}} (1,1)_{-2} \oplus (6,2)_{-1} \oplus (\mathfrak{gl}(4) \oplus \mathfrak{sl}(2)) \oplus (6,2)_{+1} \oplus (1,1)_{+2}$$

respectively, depending on the higher-dimensional origin. The shift symmetries $(6,2)_{+1}$ and $(1,1)_{+2}$ in the IIB decomposition are realized on the scalars descending from the IIB 2-forms and 4-form, respectively.

依高维起源不同分别对应。IIB 分解中的平移对称性 $(6,2)_{+1}$ 与 $(1,1)_{+2}$ 分别作用在源自 IIB 2-form 和 4-form 的标量上。

For $D = 5$ maximal supergravity [46], the geometric symmetries enhance to an $E_{6(6)}$ global symmetry group, with the decompositions (80) corresponding to

对于 $D = 5$ 维最大超引力 [46]，几何对称性增强为 $E_{6(6)}$ 整体对称群，分解式 (80) 对应于

$$\mathfrak{e}_{6(6)} \xrightarrow{\text{11D}} 1_{-2} \oplus 20_{-1} \oplus \mathfrak{gl}(6) \oplus 20_{+1} \oplus 1_{+2}, \tag{111}$$

$$\mathfrak{e}_{6(6)} \xrightarrow{\text{IIB}} (5',1)_{-2} \oplus (10,2)_{-1} \oplus (\mathfrak{gl}(5) \oplus \mathfrak{sl}(2)) \oplus \left(10',2\right)_{+1} \oplus (5,1)_{+2},$$

respectively, depending on the higher-dimensional origin. The shift symmetry 1_{+2} is realized on the scalar descending from the 11D 6-form. Recall that this form is not present in the original 11D Lagrangian (12), i.e., after reduction of (12) to $D = 5$ dimensions, the full scalar coset space sigma model $E_{6(6)}/USp(8)$ can only be realized after dualizing the 3-form $A_{\mu\upsilon\rho}$, descending from the 11D 3-form, into a scalar field (The corresponding $D = 5$ duality equation follows from dimensional reduction of (16).).

依高维起源不同分别对应。平移对称性 1_{+2} 作用在源自 11 维 6-form 的标量上。请注意该形式不出现在原始 11 维拉格朗日量 (12) 中，也就是说，将 (12) 约化到 $D = 5$ 维后，完整的标量陪集空间 sigma 模型 $E_{6(6)}/USp(8)$ 只有将源自 11 维 3-form 的 3-form $A_{\mu\upsilon\rho}$ 对偶化为标量场才能实现 (对应的 $D = 5$ 维对偶方程可由式 (16) 维度约化得到)。

We refer to [47] for a systematic discussion of the maximal supergravities in various dimensions, together with their symmetries and their eleven-dimensional origin.

关于各维最大超引力及其对称性和十一维起源的系统讨论可见文献 [47]。

$D = 4$ Maximal Supergravity

$D = 4$ maximal 超引力

Let us discuss in a little more detail the case of $D = 4$ maximal supergravity. Historically, this was the first example of exceptional symmetry groups appearing in supergravity theories and playing an essential role in their explicit construction. The field content of this theory is the massless $\mathcal{N} = 8$ supergravity multiplet

我们来更详细地讨论 $D=4$ 极大超引力的情形。从历史上看，这是例外对称群首次出现在超引力理论中并在其显式构造中发挥核心作用的例子。该理论的场内容是无质量 $\mathcal{N}=8$ 超引力多重态

$$\{g_{\mu\nu}, \psi_\mu{}^i, A_\mu{}^\Lambda, \chi^{ijk}, \phi^{ijkl}\},\ i=1,\dots,8,\ \Lambda=1,\dots,28, \tag{112}$$

which comprises the graviton, 8 gravitinos, 28 vector fields, 56 spin-1/2 fermions, and 70 scalar fields. The complete theory was obtained in [6] by dimensional reduction of the 11D supergravity on a seven-torus T^7 and realizing the exceptional symmetry group $\mathrm{E}_{7(7)}$. As discussed in section "Reduction of Pure Gravity", the reduction of pure gravity from eleven dimensions down to $D=4$ dimensions yields a gravitational theory with seven Abelian vector fields $A_\mu{}^n, n=1,\dots,7$, and $1+27$ scalar fields, parametrizing the coset space GL(7)/SO(7). The dimensional reduction of the antisymmetric 3-form to $D=4$ dimensions as described in section "Reduction of p -Forms" gives rise to one 3-form field, seven 2-form fields, $\begin{pmatrix} 7 \\ 2 \end{pmatrix} = 21$ vectors, and additional $\begin{pmatrix} 7 \\ 3 \end{pmatrix} = 35$ scalar fields. A priori, the field content thus looks quite different from the $\mathcal{N}=8$ multiplet (112). Including the (normalized) GL(1) charges from (52), (62), we find the four-dimensional bosonic field content

它包含引力子、8 个引力微子、28 个矢量场、56 个自旋 1/2 费米子和 70 个标量场。完整理论是文献 [6] 中通过将 11 维超引力在七环面 T^7 上维度约化，并实现例外对称群 $\mathrm{E}_{7(7)}$ 得到的。正如“纯引力的约化”一节所述，将纯引力从 11 维约化到 $D=4$ 维，会得到一个包含 7 个阿贝尔矢量场 $A_\mu{}^n, n=1,\dots,7$ 以及 1+27 个标量场的引力理论，这些标量参数化陪集空间 GL(7)/SO(7)。正如“p -形式的约化”一节所述，将反对称 3-form 维度约化到 $D=4$ 维，会得到一个 3-form 场、7 个 2-form 场、$\begin{pmatrix} 7 \\ 2 \end{pmatrix} = 21$ 个矢量和额外的 $\begin{pmatrix} 7 \\ 3 \end{pmatrix} = 35$ 个标量。先验来看，此时的场内容和 $\mathcal{N}=8$ 多重态 (112) 差异很大。纳入 (52)、(62) 中归一化的 GL(1) 荷后，我们得到四维玻色场内容为

$$g_{\mu\nu} : 1_0$$

$$\phi : 1_0 + 27_0 + 35_{-2}$$

$$A_\mu : 7'_{+3} + 21_{+1}$$

$$A_{\mu\upsilon} : 7_{+4},$$

$$A_{\mu\upsilon\rho} : 1_{+7}, \tag{113}$$

with the fields (other than the GL(7)/SO(7) scalars) falling into linear GL(7) representations. Reduction of the 11D Lagrangian (12) yields the sum of (54) and (74), together with the reduction of the 11D CS term. The symmetries of the resulting action span the algebra

其中除 GL(7)/SO(7) 标量外的所有场都属于 GL(7) 的线性表示。约化 11 维拉氏量 (12) 得到 (54) 与 (74) 之和，同时包含 11 维陈-西蒙项的约化结果。所得作用量的对称性张成以下代数

$$\mathfrak{gl}(7) \oplus 35'_{+2} \,, \tag{114}$$

where the $35'_{+2}$ generators induce the shift symmetries (65) on the scalars descending from the 11D 3-form. In order to make contact with the $\mathcal{N}=8$ supergravity multiplet, and realize the symmetry enhancement (80), we need to first dualize the 2- forms $A_{\mu\nu m}$ into scalar fields (put equivalently, we trade them for the corresponding scalars $A_{m_1\ldots m_6}$ descending from the dual 11D 6-form). With the associated shift symmetries (65), the algebra of manifest global symmetries enhances to (66)

其中 $35'_{+2}$ 生成元对源自 11 维 3-form 的标量诱导出平移对称性 (65)。为了和 $\mathcal{N}=8$ 超引力多重态对应，并实现对称性增强 (80)，我们首先需要将 2-form 场 $A_{\mu\nu m}$ 对偶化为标量场 (等价地，我们将其替换为源自对偶 11 维 6-form 的对应标量 $A_{m_1\ldots m_6}$)。结合相关的平移对称性 (65)，显式整体对称的代数增强为 (66)

$$\mathfrak{gl}(7) \oplus 35'_{+2} \oplus 7_{+4} \tag{115}$$

The nontrivial algebra structure of the charged generators directly descends from the 11D gauge algebra (21).

带荷生成元的非平凡代数结构直接源自 11 维规范代数 (21)。

The dynamics of the resulting set of 70 scalar fields is described by a sigma model on the coset space

最终得到的这 70 个标量场的动力学由陪集空间上的 sigma 模型描述

$$\mathrm{G/K} = \mathrm{E}_{7(7)}/\mathrm{SU}(8) \supset \mathrm{GL}(7)/\mathrm{SO}(7) \,, \tag{116}$$

according to (87). The symmetry enhancement from (115) to $\mathfrak{e}_{7(7)}$ is realized along the lines discussed in the previous sections: decomposing $\mathrm{E}_{7(7)}$ according to its $\mathfrak{gl}(1)$ grading, the algebra splits into the form (80)

符合 (87) 式。对称性从 (115) 增强到 $\mathfrak{e}_{7(7)}$ 的实现方式符合前几节的讨论: 将 $\mathrm{E}_{7(7)}$ 按其 $\mathfrak{gl}(1)$ 分次分解，代数分裂为 (80) 的形式

$$\mathfrak{e}_{7(7)} \overset{11D}{\to} 7'_{-4} \oplus 35_{-2} \oplus \mathfrak{gl}(7) \oplus 35'_{+2} \oplus 7_{+4}, \tag{117}$$

with the generators of non-negative charge spanning (115). The negative grading generators correspond to the hidden symmetries that are present after reduction to four dimensions but have no manifest origin in 11D supergravity. An explicit parametrization of the coset space (116) in terms of the 11D fields is given in the triangular gauge (91) as

非负荷的生成元张成 (115)。负分次生成元对应约化到四维后存在，但在 11 维超引力中没有显式起源的隐藏对称性。陪集空间 (116) 用 11 维场的显式参数化可在三角规范 (91) 下写为

$$\mathcal{V} \equiv \exp\left[\varepsilon^{klmnpqr} A_{klmnpq} t_{(+4)r}\right] \exp\left[A_{kmn} t^{kmn}_{(+2)}\right] V_{\mathrm{GL}(7)}. \tag{118}$$

Here, $V_{\mathrm{GL}(7)} \in \mathrm{GL}(7)$ is the internal block of the 11D metric (up to some power of its determinant), whereas the $t_{(+n)}$ refer to the $\mathrm{E}_{7(7)}$ generators of positive grading in (117). For convenience, all generators are evaluated in the fundamental 56 representation. Under the global symmetry $\mathfrak{e}_{7(7)}$, the vielbein transforms as (90), inducing a nonlinear action on the scalar fields. The bosonic sector of the theory can be formulated in terms of the symmetric, positive definite matrix

在此处，$V_{\mathrm{GL}(7)} \in \mathrm{GL}(7)$ 是 11 维度规的内部块 (直到其行列式的某次幂)，而 $t_{(+n)}$ 对应式 (117) 中正分次的 $\mathrm{E}_{7(7)}$ 生成元。为方便起见，所有生成元均取基础 56 表示计算。在整体对称性 $\mathfrak{e}_{7(7)}$ 下，标架按式 (90) 变换，从而在标量场上诱导出非线性作用。该理论的玻色子 sector 可以用对称正定矩阵表述

$$\mathcal{M}_{MN} = \mathcal{V}_M{}^{\underline{A}}\, \mathcal{V}_N{}^{\underline{A}}, \tag{119}$$

on which $\mathfrak{e}_{7(7)}$ acts by conjugation. In particular, the coset space sigma model (87) can be written as

$\mathfrak{e}_{7(7)}$ 通过共轭作用作用于该矩阵。具体而言，陪集空间 sigma 模型 (87) 可以写为

$$\mathcal{L} = \frac{1}{8}\, |e|\, \partial_\mu \big(\mathcal{M}^{-1}\big)^{MN} \partial^\mu \mathcal{M}_{MN}. \tag{120}$$

In order to realize the global $\mathrm{E}_{7(7)}$ symmetry on the vector fields, we have to combine the 28 vector fields of (113) with their magnetic duals into the irreducible 56 of $\mathrm{E}_{7(7)}$

为了在矢量场上实现整体 $\mathrm{E}_{7(7)}$ 对称性，我们需要将式 (113) 的 28 个矢量场与其磁对偶结合，组成 $\mathrm{E}_{7(7)}$ 的不可约 56 表示

$$56 \to 7_{-3} + 21'_{-1} + 21_{+1} + 7'_{+3}. \tag{121}$$

Once more, this illustrates that the realization of the full enhanced symmetry group involves the original and the dual fields of the theory. The dynamics of the vector fields is described by the $\mathrm{E}_{7(7)}$ covariant twisted self-duality equation

这再次说明，要实现完整的增强对称群，需要同时用到理论的原始场和对偶场。矢量场的动力学由 $\mathrm{E}_{7(7)}$ 协变扭曲自对偶方程描述

$$F_{\mu\nu}{}^M = -\frac{1}{2}\, |e|\, \varepsilon_{\mu\nu\rho\sigma} \Omega^{MN} \mathcal{M}_{NK} F^{\rho\sigma K}, \tag{122}$$

for the abelian field strengths $F_{\mu\nu}{}^M = 2\partial_{[\mu} A_{\nu]}{}^M$, and $\mathcal{M}_{NK}$ from (119). Here, Ω^{MN} is the constant antisymmetric $\mathrm{E}_{7(7)}$ -invariant symplectic tensor, which establishes the embedding $\mathrm{E}_{7(7)} \subset \mathrm{Sp}(28, 28)$. In particular, the relation

该方程针对阿贝尔场强 $F_{\mu\nu}{}^M = 2\partial_{[\mu} A_{\nu]}{}^M$ 和式 (119) 中的 $\mathcal{M}_{NK}$ 给出。此处，Ω^{MN} 是满足 $\mathrm{E}_{7(7)}$ 不变性的常数反对称辛张量，给出了嵌入 $\mathrm{E}_{7(7)} \subset \mathrm{Sp}(28, 28)$ 。具体而言，关系

$$\Omega^{KL} \mathcal{M}_{LM} \Omega^{MN} \mathcal{M}_{NP} = -\delta^K{}_P, \tag{123}$$

is necessary for consistency of (122). The full bosonic sector of maximal $D = 4$ supergravity can be compactly described by a pseudo-Lagrangian

是式 (122) 自洽性的必要条件。最大 $D = 4$ 超引力的完整玻色子 sector 可以用伪拉格朗日量简洁地描述

$$\mathcal{L}_{D=4} = |e|\left(R + \frac{1}{48}\partial_\mu(\mathcal{M}^{-1})^{MN}\partial^\mu\mathcal{M}_{MN} - \frac{1}{8}\mathcal{M}_{MN}F^{\mu\nu M}F_{\mu\nu}{}^{N}\right), \tag{124}$$

combined with the twisted self-duality equation (122). Both the Lagrangian (124) and Equation (122) are manifestly $E_{7(7)}$ covariant. However,(124) yields only a pseudo-Lagrangian of the theory in that the twisted self-duality Equation (122) does not follow from the variational principle but has to be imposed separately. Only its derivative coincides with the second-order equation for the vector fields that is obtained by variation of (124).

并结合扭曲自对偶方程 (122)。拉格朗日量 (124) 和方程 (122) 都具有明显的 $E_{7(7)}$ 协变性。然而, (124) 仅给出该理论的伪拉格朗日量: 扭曲自对偶方程 (122) 无法从变分原理导出，必须额外单独施加。只有该方程的导数与变分 (124) 得到的矢量场二阶方程一致。

A standard action principle of the theory can only be spelled out after sacrificing the manifest $E_{7(7)}$ invariance and splitting the 56 vector fields into 28 + 28 as

只有放弃显式的 $E_{7(7)}$ 不变性，将 56 个矢量场拆分为 28 + 28 ，才能写出该理论的标准作用量原理，即

$$A_\mu{}^M = \{A_\mu{}^\Lambda, A_{\mu\Lambda}\}. \tag{125}$$

An action can then be constructed in terms of half of the fields $A_\mu{}^A$ considered as independent propagating (electric) fields, while the $A_{\mu\Lambda}$ are defined via (122) as their on-shell (magnetic) duals [48]. This is achieved by replacing the Maxwell term of (124) by the Lagrangian

随后可以将一半的场 $A_\mu{}^A$ 视为独立传播的 (电) 场，构造出作用量，而 $A_{\mu\Lambda}$ 则通过式 (122) 定义为其在壳 (磁) 对偶 [48]。具体做法是将式 (124) 的麦克斯韦项替换为如下拉格朗日量

$$\mathcal{L}_{\text{vector}} = \frac{1}{4}|e|\,\mathcal{I}_{\Lambda\Sigma}F_{\mu\nu}{}^{\Lambda}F^{\mu\nu\Sigma} + \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\mathcal{R}_{\Lambda\Sigma}F_{\mu\nu}{}^{\Lambda}F_{\rho\sigma}{}^{\Sigma}, \tag{126}$$

in terms of the 28 abelian field strengths $F_{\mu\nu}{}^\Lambda$, with the symmetric kinetic matrices $\mathcal{I}_{\Lambda\Sigma}$ and $\mathcal{R}_{\Lambda\Sigma}$ related to the matrix $\mathcal{M}$ from (119) as

该拉格朗日量用 28 个阿贝尔场强 $F_{\mu\nu}{}^\Lambda$ 表示，对称动能矩阵 $\mathcal{I}_{\Lambda\Sigma}$ 和 $\mathcal{R}_{\Lambda\Sigma}$ 与式 (119) 中的矩阵 $\mathcal{M}$ 满足关系

$$\mathcal{M} \equiv -\begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix}, \tag{127}$$

in the split (125). In particular, $\mathcal{I}_{\Lambda\Sigma}$ is negative definite, such that the kinetic term in (126) comes with the correct sign. This yields a true action principle for the bosonic sector of $D = 4$ maximal supergravity, whose off-shell symmetry however is reduced to a subgroup of $E_{7(7)}$, whereas the full $E_{7(7)}$ global symmetry is realized only on-shell. A similar pattern is observed in all even-dimensional maximal supergravities in $D < 10$.

在分拆 (125) 中。特别地，$\mathcal{I}_{\Lambda\Sigma}$ 是负定的，因此 (126) 中的动能项符号正确。这为 $D = 4$ 极大超引力的玻色子 sector 给出了真正的作用量原理，但其离壳对称性约化为 $E_{7(7)}$ 的一个子群，而完整的 $E_{7(7)}$ 整体对称性仅在壳上实现。在 $D < 10$ 中所有偶数维极大超引力里都观测到了相同的规律。

Different choices for the electric/magnetic split (125) correspond to different electric frames and can be related by symplectic rotation. These give rise to different off-shell formulations of the theory. In particular, the off-shell symmetry group depends on the particular choice of the electric frame. Choosing the 28 electric vectors to be

电/磁分拆 (125) 的不同选择对应不同的电框架，可通过辛旋转相互联系。它们会给出该理论不同的离壳表述。具体来说，离对称群依赖于电框架的具体选择。选取 28 个电矢量为

$$7'_{+3} + 21'_{-1} \;:\; A_\mu^\Lambda = \left\{ A_\mu{}^m, A_\mu{}^{mn} \right\}, \tag{128}$$

among the 56 vectors (121), the symmetry of the Lagrangian (126) is given by the electric subgroup $SL(8) \subset E_{7(7)}$, with the vectors (128) spanning its irreducible 28-dimensional representation.

在 56 个矢量 (121) 中，拉格朗日量 (126) 的对称性由电子群 $SL(8) \subset E_{7(7)}$ 给出，矢量 (128) 张成该群的不可约 28 维表示。

Finally, the 3-form in (113) is non-propagating and can be consistently set to zero in the dimensional reduction. Strictly speaking, however, its field equations only imply that its field strength is constant and may be set to an arbitrary value. Keeping this integration constant produces a one-parameter deformation of the maximally supersymmetric theory [49]. It breaks the global $E_{7(7)}$ symmetry, and closer inspection shows that this integration constant is only one component of an irreducible 912-dimensional representation $E_{7(7)}$ [50]. Switching on other parameters within this representation leads to different maximally supersymmetric theories, which generically have non-Abelian gauge groups and matter charged under the gauge group; these deformations are the so-called gauged supergravities and may correspond to more complicated compactifications in the presence of background fluxes and/or geometric fluxes (see [51] for a review). In particular, these theories include the compactification of eleven-dimensional supergravity on the seven-sphere S^7 , which gives rise to a four-dimensional theory with compact non-Abelian gauge group SO(8) [52].

最后，(113) 中的 3-形式是非传播的，可以在维约化中一致地设为零。但严格来说，它的场方程仅要求其场强为常数，可以取任意值。保留该积分常数会得到极大超对称理论的单参数变形 [49]。它会破缺整体 $E_{7(7)}$ 对称性，进一步研究表明该积分常数只是不可约 912 维表示 $E_{7(7)}$ 的一个分量 [50]。开启该表示内的其他参数会得到不同的极大超对称理论，这些理论一般具有非阿贝尔规范群以及在规范群下带荷的物质；这类变形就是所谓的规范化超引力，可能对应背景通量和/或几何通量存在时更复杂的紧致化 (综述见 [51])。特别地，这些理论包含十一维超引力在七球面 S^7 上的紧致化，由此得到具有紧致非阿贝尔规范群 SO(8) 的四维理论 [52]。

Let us finally note that maximal $D = 4$ supergravity can of course also be obtained by reduction of the IIB theory on a six-torus T^6 . In this case, the decomposition (80) of the algebra is given by

最后我们指出，极大 $D=4$ 超引力当然也可以通过 IIB 理论在六环面 T^6 上约化得到。在这种情况下，代数的分解式 (80) 为

$$\mathfrak{e}_{7(7)} \xrightarrow{\text{IIB}} (1,2)_{-3} \oplus \left(15',1\right)_{-2} \oplus (15,2)_{-1} \oplus (\mathfrak{gl}\,(6) \oplus \mathfrak{sl}\,(2))$$

$$\oplus \left(15',2\right)_{+1} \oplus (15,1)_{+2} \oplus (1,2)_{+3}, \tag{129}$$

and the shift symmetries $\left(15',2\right)_{+1}, (15,1)_{+2}$, and $(1,2)_{+3}$ are realized on the scalars descending from the IIB 2-forms, 4-form and dual 6-forms, respectively.

而位移对称性 $\left(15',2\right)_{+1}, (15,1)_{+2}$ 和 $(1,2)_{+3}$ 分别作用在来自 IIB 理论 2-形式、4-形式和对偶 6-形式的标量上。

Lower-Dimensional Supergravities

低维超引力

The symmetry enhancement and appearance of exceptional symmetry groups continues and becomes even more intricate with maximal supergravities in lower dimensions. For $D=3$ maximal supergravity [53], the geometric symmetries enhance to an $\mathrm{E}_{8(8)}$ global symmetry group, with the decomposition (80) corresponding to

对称性增强和例外对称群的出现持续存在，在低维最大超引力中变得更为 intricate。对于 $D=3$ 最大超引力 [53]，几何对称性增强为一个 $\mathrm{E}_{8(8)}$ 整体对称群，其分解式 (80) 对应于

$$\mathfrak{e}_{8(8)} \xrightarrow{\text{11D}} 8_{-3} \oplus 28'_{-2} \oplus 56_{-2} \oplus \mathfrak{gl}\,(8) \oplus 56'_{+1} \oplus 28_{+2} \oplus 8'_{+3}. \tag{130}$$

The shift symmetries $56'_{+1}$ and 28_{+2} are realized on the scalars descending from the 11D 3-form and dual 6-form, respectively. A new feature arising in the reduction to $D=3$ dimensions is the fact that the realization of the full symmetry algebra requires the dualization of the 8 Kaluza-Klein vector fields ${A_\mu}^m$ (45) into scalar fields ϕ_m . We have already encountered this in the example of the SL(2) Ehlers symmetry (100), which is embedded as a subgroup into the $\mathrm{E}_{8(8)}$ of (130). The dual scalar fields encountered in $D>3$ supergravities have all been identified among the components of the 11D dual 6-form. In contrast, the higher-dimensional interpretation of the ϕ_m is more subtle, as they should be identified with components of the 11D ”dual graviton” [54, 55] , whose proper definition beyond the linearized theory and before dimensional reduction remains ambiguous. The shift symmetries $8'_{+3}$ in (130) act on these dual scalars ϕ_m , and the full bosonic sector of the theory is given by a gravity coupled sigma model on the coset space $\mathrm{E}_{8(8)}/\mathrm{SO}\,(16)$. For completeness, let us also note the decomposition (80) of $\mathrm{E}_{8(8)}$ w.r.t. IIB supergravity

位移对称性 $56'_{+1}$ 和 28_{+2} 分别作用于来自 11 维 3 形式和对偶 6 形式的标量场。约化到 $D = 3$ 维时出现的一个新特征是: 要实现完整对称代数，需要将 8 个卡鲁扎-克莱因矢量场 $A_\mu{}^m$ (45) 对偶化为标量场 ϕ_m 。我们已经在 SL(2) 埃勒斯对称性 (100) 的例子中遇到过这种情况，该对称性作为子群嵌入 (130) 的 $\mathrm{E}_{8(8)}$ 中。$D > 3$ 超引力中的对偶标量场都已被对应到 11 维对偶 6 形式的分量中。与之相对，ϕ_m 的高维解释更为微妙，因为它们应当对应于 11 维“对偶引力子”[54, 55] 的分量，而该引力子在线性化理论之外、维约化之前的恰当定义仍然不明确。(130) 中的位移对称性 $8'_{+3}$ 作用在这些对偶标量 ϕ_m 上，理论的完整玻色子部分由耦合到引力、定义在陪集空间 $\mathrm{E}_{8(8)}/\mathrm{SO}(16)$ 上的 sigma 模型给出。为完整起见，我们还要说明 $\mathrm{E}_{8(8)}$ 相对于 IIB 超引力的分解式 (80)

$$\mathfrak{e}_{8(8)} \stackrel{\text{IIB}}{\to} (7,1)_{-4} \oplus (7',2)_{-3} \oplus \left(35',1\right)_{-2} \oplus (21,2)_{-1} \oplus (\mathfrak{gl}(7) \oplus \mathfrak{sl}(2))$$

$$\oplus\left(21',2\right)_{+1} \oplus (35,1)_{+2} \oplus (7,2)_{+3} \oplus (7',1)_{+4}. \tag{131}$$

In the reduction to $D = 2$, the structures become even richer. Extrapolating the exceptional series of Lie algebras with Dynkin diagram given by Fig. 1 defines the infinite-dimensional algebra $\mathfrak{e}_{9(9)}$ as the (centrally extended) loop algebra $\widehat{\mathfrak{e}_{8(8)}}$. This algebra naturally acts on infinite-dimensional representations, which are built by the infinite towers of dual scalar fields that are defined on-shell by repeatedly dualizing the 128 scalar fields that appear in the reduction of 11D supergravity. These physical scalars together with the infinite tower of dual potentials can be cast into the coset space $\mathrm{E}_{9(9)}/\mathrm{K}(\mathrm{E}_9)$ with a well-defined action of the $\mathfrak{e}_{9(9)}$ symmetry algebra. We refer to [17, 56 − 61] for details.

约化到 $D = 2$ 时，结构变得更加丰富。外推具有图 1 所述邓肯图的例外李代数序列，可定义无穷维代数 $\mathfrak{e}_{9(9)}$ 为 (中心扩张的) 圈代数 $\widehat{\mathfrak{e}_{8(8)}}$ 。该代数自然作用在无穷维表示上，这些表示由无穷层对偶标量场构建而来；而这些对偶标量场是通过对 11 维超引力约化中出现的 128 个标量场反复对偶化，得到的在壳场。这些物理标量连同无穷层对偶势可以被整理进陪集空间 $\mathrm{E}_{9(9)}/\mathrm{K}(\mathrm{E}_9)$ ，其中 $\mathfrak{e}_{9(9)}$ 对称代数具有良定义的作用。详情参见 [17, 56 − 61] 。

The large global symmetry groups appearing in low dimensions have been of practical use in order to generate solutions of the higher-dimensional theories. While we have seen that part of the global symmetries (66) after toroidal reduction descends from the action of particular higher-dimensional diffeomorphisms and gauge transformations, the hidden symmetries combined in the algebra $\mathfrak{n}_-$ in (80) do not have a direct higher-dimensional interpretation. A solution of the higher-dimensional field equations with a sufficient number of commuting Killing vector fields induces a solution of the lower-dimensional theory on which the action of the full symmetry group associated with (80) can be explicitly computed. Lifting the result back to higher dimensions then produces a genuinely new solution to the higher-dimensional field equations. It is in this context of solution generating methods in Einstein gravity that hidden symmetries have first been discovered in $D = 3$ reductions [42,62-64], as well as in the infinite-dimensional case in $D = 2$ reductions [65-69] (see [70] for a review). The larger the group of hidden symmetries, the larger is the orbit of newly generated solutions. In supergravity, the hidden symmetries from $D = 3$ reductions have been exploited as solution generating techniques for the construction of black hole and black ring solutions in higher dimensions (see, e.g., [71-73]).

低维出现的大整体对称群在生成高维理论的解时具有实际用途。我们已经看到，环面约化后的整体对称性 (66) 有一部分源自特定高维微分同胚和规范变换的作用，而 (80) 中合并为代数 $\mathfrak{n}_-$ 的隐对称并没有直接的高维解释。一个带有足够多对易基灵矢量场的高维场方程解，会诱导出低维理论的一个解，在此低维解上可以显式计算 (80) 关联的完整对称群的作用。将结果提升回高维后，就会得到高维场方程的一个全新解。隐对称最早就是在爱因斯坦引力的解生成方法这一背景下，于 $D = 3$ 约化 [42,62-64] 以及无限维情形下的 $D = 2$ 约化 [65-69] 中被发现的 (综述见 [70])。隐对称群越大，新生成解的轨道也就越大。在超引力中，来自 $D = 3$ 约化的隐对称已被用作解生成技术，用于构造高维下的黑洞与黑环解 (参见例如 [71-73])。

Let us finally mention that the extrapolation to yet higher $d > 9$ leads to the overextended and very-extended Kac-Moody algebras $\mathfrak{e}_{10}$ and $\mathfrak{e}_{11}$, respectively, which each have been conjectured in different context to play a fundamental role in the full 11D supergravity [55,74].

最后我们要提到，向更高阶 $d > 9$ 外推会得到超扩张与极扩张卡茨-穆迪代数 $\mathfrak{e}_{10}$ 和 $\mathfrak{e}_{11}$ ，二者分别在不同背景下被猜想在完整 11 维超引力中发挥基础性作用 [55,74]。

Exceptional Field Theory

例外场论

We have seen in the preceding sections that the global symmetry groups of lower-dimensional maximal supergravity theories are only partially explained by the diffeomorphism and gauge symmetries of 11D supergravity. In particular, after dimensional reduction of (12), it is only after dualization of some of the lower-dimensional fields that the full global symmetry group $\mathrm{E}_{d(d)}$ becomes manifest. In this final section, we briefly review the reformulation of 11D supergravity as an exceptional field theory (ExFT) [21]. As an illustration, we focus on the example of $\mathrm{E}_{7(7)}$ ExFT [75]. In this formulation, dimensional reduction of the higher-dimensional theory directly leads to the Lagrangian (124) and equations of motion (122), in which the global exceptional symmetry $\mathrm{E}_{7(7)}$ is manifest.

我们在前文章节中已经看到，低维最大超引力理论的整体对称性仅能由 11 维超引力的微分同胚和规范对称性部分解释。具体而言，对 (12) 做维约化后，只有将部分低维场对偶化，完整整体对称群 $\mathrm{E}_{d(d)}$ 才会显现。在最后这一节中，我们简要回顾将 11 维超引力重新表述为例外场论 (ExFT)[21] 的工作。为便于说明，我们重点讨论 $\mathrm{E}_{7(7)}$ 例外场论的例子 [75]。在该表述下，高维理论的维约化会直接得到拉格朗日量 (124) 和运动方程 (122)，其中整体例外对称性 $\mathrm{E}_{7(7)}$ 是显然的。

Starting from 11D supergravity, we may perform a decomposition of fields (45) and (60), as appropriate for dimensional reduction on T^7 , and rewrite the theory in terms of the various components, without however imposing (43), i.e., keeping the full eleven-dimensional coordinate dependence of all fields. This is akin to a Kaluza-Klein compactification of the 11D theory in which all the Kaluza-Klein towers of massive fields are kept. From the four-dimensional point of view, this corresponds to a theory with infinitely many fields, compactly encoded in the dependence of all fields on the internal coordinates y^m . As a standard structure of Kaluza-Klein theory, the resulting theory comes with an infinite-dimensional non-Abelian gauge structure, corresponding to the diffeomorphisms and gauge symmetries on the internal space. After dualization of fields, following the steps of section " $D = 4$ Maximal Supergravity" (In the reduction to $D = 4$ discussed in section " $D = 4$ Maximal Supergravity", dualization always refers to abelian vectors and p -forms. Here, the non-Abelian structure related to the dependence on internal coordinates does not pose an obstruction to the dualization but can be compensated by Stückelberg-type couplings to higher degree forms, as is common in gauged supergravity [51, 76]. In particular, this is the underlying reason for extending the bosonic field content to (133).), this leads to a formulation of the bosonic field content in terms of the $\mathrm{E}_{7(7)}$ fields

从 11 维超引力出发，我们可以对场做分解 (45) 和 (60)——这一分解适合在 T^7 上做维约化，再将理论按不同分量改写，且不施加条件 (43)，即保留所有场完整的 11 维坐标依赖。这类似于 11 维理论的卡鲁扎-克莱因紧化，其中保留了所有有质量场的卡鲁扎-克莱因塔。从四维视角来看，这对应一个有无穷多场的理论，所有场对内部坐标 y^m 的依赖就紧凑地编码了这些场。作为卡鲁扎-克莱因理论的标准结构，所得理论带有一个无穷维非阿贝尔规范结构，对应内部空间上的微分同胚和规范对称性。在依照“$D = 4$ 最大超引力”一节的步骤完成场对偶化后 (在“$D = 4$ 最大超引力”一节讨论的约化到 $D = 4$ 的过程中，对偶化始终指对阿贝尔矢量和 p -形式做对偶。此处，与内部坐标依赖相关的非阿贝尔结构不会对偶化造成阻碍，它可以通过施蒂克尔贝格型耦合补偿到高次形式，这在定规范超引力中是常见的 [51, 76]。这一点正是将玻色场内容扩展为 (133) 的根本原因。)，我们最终得到玻色场内容按 $\mathrm{E}_{7(7)}$ 场表述的形式

$$\left\{g_{\mu\nu}, \mathcal{M}_{MN}, \mathcal{A}_{\mu}{}^{M}\right\},\ \mu, \nu = 0, \dots, 3,\ M = 1, \dots, 56, \tag{132}$$

in alignment with the field content of $D = 4$ maximal supergravity, except for all fields keeping their dependence on the internal coordinates y^m . In particular, $\mathcal{M}_{MN}$ still is a matrix of type (119), representing the coset space $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$. Its vielbein (119) is parametrized as (118) in terms of the 11D fields. On top of these fields, the formulation requires 2-forms of the type

与 $D = 4$ 最大超引力的场内容一致，区别仅在于所有场都保留了对内部坐标 y^m 的依赖。具体来说，$\mathcal{M}_{MN}$ 仍然是 (119) 型的矩阵，代表陪集空间 $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ 。它的 vielbein(119) 由 11 维场按 (118) 参数化。除这些场之外，该表述还需要如下类型的 2-形式

$$\left\{\mathcal{B}_{\mu\nu\alpha}, \mathcal{B}_{\mu\nu M}\right\},\ \alpha = 1, \dots, 133, \tag{133}$$

also depending on all coordinates. Here, α is an index in the adjoint representation of $\mathrm{E}_{7(7)}$, while the 562-forms $\mathcal{B}_{\mu\nu M}$ satisfy algebraic constraints

同样依赖所有坐标。此处，α 是 $\mathrm{E}_{7(7)}$ 伴随表示的一个指标，而 56 个 2-形式 $\mathcal{B}_{\mu\nu M}$ 满足代数约束

$$(t_\alpha)_K{}^M \Omega^{NK} \mathcal{B}_{\mu\nu M} \mathcal{B}_{\rho\sigma N} = 0 = \Omega^{MN} \mathcal{B}_{\mu\nu M} \mathcal{B}_{\rho\sigma N}, \tag{134}$$

where $(t_\alpha)_K{}^M$ denote the generators of the algebra $\mathfrak{e}_{7(7)}$.

其中 $(t_\alpha)_K{}^M$ 表示代数 $\mathfrak{e}_{7(7)}$ 的生成元。

The 11D field equations can be written in terms of the objects (132), (133). Remarkably, the resulting equations can equivalently be derived from first principles based on the infinite-dimensional gauge structure of the theory. To this end, the internal coordinates are embedded into an extended spacetime with coordinates transforming in the fundamental 56 of $\mathrm{E}_{7(7)}$. The original physical coordinates are recovered as the solution of an $E_{7(7)}$ -covariant section constraint. On the extended spacetime, the original diffeomorphisms and gauge symmetries are unified into generalized diffeomorphisms [77-86], which provide the organizing structure for the construction of the theory.

11 维场方程可以通过 (132)、(133) 中的对象写出。值得注意的是，所得方程也可以从该理论无穷维规范结构的第一性原理推导得到。为此，内部坐标被嵌入一个扩展时空，其坐标按 $E_{7(7)}$ 的基础 56 表示变换。原物理坐标可以通过满足一个 $E_{7(7)}$ 协变的截面约束得到。在扩展时空上，原微分同胚和规范对称性被统一为广义微分同胚 [77-86]，这为理论构造提供了组织框架。

Specifically, the action of generalized diffeomorphisms on the scalar matrix $\mathcal{M}_{MN}$ is of the form [81, 82]

具体而言，广义微分同胚对标量矩阵 $\mathcal{M}_{MN}$ 的作用具有 [81, 82] 的形式

$$\delta_\Lambda \mathcal{M}_{MN} = \mathcal{L}_\Lambda \mathcal{M}_{MN} = \Lambda^K \partial_K \mathcal{M}_{MN} + 24 \partial_L \Lambda^K \mathbb{P}^K{}_L{}^P{}_{(M,M)P}. \tag{135}$$

Here, $\mathbb{P}^K{}_L{}^P{}_M$ is the projector on the adjoint representation of $E_{7(7)}$, which takes the explicit form (For the raising and lowering of symplectic indices, we use North-West South-East conventions, i.e., $Z^M = \Omega^{MN} Z_N$ and $Z_M = Z^N \Omega_{NM}$.)

此处，$\mathbb{P}^K{}_L{}^P{}_M$ 是 $E_{7(7)}$ 伴随表示的投影算子，其具体形式为 (对于辛指标的升降，我们采用西北-东南约定，即 $Z^M = \Omega^{MN} Z_N$ 和 $Z_M = Z^N \Omega_{NM}$ 。)

$$\mathbb{P}^K{}_M{}^L{}_N = (t_\alpha)_M{}^K (t^\alpha)_N{}^L \tag{136}$$

$$= \frac{1}{24} \delta_M{}^K \delta_N{}^L + \frac{1}{12} \delta_M{}^L \delta_N{}^K + (t_\alpha)_{MN} (t^\alpha)^{KL} - \frac{1}{24} \Omega_{MN} \Omega^{KL}.$$

The 56 components of the gauge parameter Λ^M in (135) combine the gauge symmetries associated with the vector fields (121), i.e., $7 + 21$ of the gauge symmetries descend from the 11D diffeomorphisms and tensor gauge transformations, respectively, while the other half is associated with the dual vector fields.

(135) 中规范参数 Λ^M 的 56 个分量合并了与矢量场 (121) 相关的规范对称性，即分别有 $7 + 21$ 的规范对称性源自 11 维微分同胚和张量规范变换，另一半则与对偶矢量场相关。

The coordinate dependence of all fields and gauge parameters is constrained by the so-called section constraint. The latter imposes an embedding of the physical coordinates $\partial_m \hookrightarrow \partial_M$ imposing that every couple of fields (Φ_1, Φ_2) satisfies

所有场和规范参数的坐标依赖受所谓截面约束的限制。该约束要求对物理坐标 $\partial_m \hookrightarrow \partial_M$ 做嵌入，使得任意一对场 (Φ_1, Φ_2) 都满足

$$(t_\alpha)_K{}^M \Omega^{NK} \partial_M \Phi_1 \partial_N \Phi_2 = 0 = \Omega^{MN} \partial_M \Phi_1 \partial_N \Phi_2, \tag{137}$$

similar to (134). This is an $E_{7(7)}$ -covariant way of stating that only 7 out of the formally 56 derivatives ∂_M appearing in (135) actually have a nontrivial action, as compatible with the eleven-dimensional nature of the original theory. It is straightforward to check that the algebra of generalized diffeomorphisms (135) only closes under the assumption of (137).

与 (134) 类似。这是一种 $E_{7(7)}$ 协变的表述，说明 (135) 中形式上的 56 个导数 ∂_M 只有 7 个具有非平凡作用，这与原始理论的十一维性质相容。不难验证，广义微分同胚代数 (135) 仅在 (137) 的假设下才封闭。

Not unexpectedly, there are in fact two inequivalent maximal solutions to (137) that restrict the dependence of all fields to 7 and 6 coordinates, respectively. They correspond to 11D and IIB supergravity, respectively, which are thus both embedded into the same exceptional field theory. Specifically, the solutions to the section constraint (137) are based on the decompositions (117) and (129), respectively, of $\mathfrak{e}_{7(7)}$ and correspond to the embedding of internal coordinates $\partial_m \hookrightarrow \partial_M$ realized according to (The fact that these embeddings provide solutions to (137) can immediately be inferred from the $\mathfrak{gl}(1)$ gradings.)

不出所料，方程 (137) 实际上存在两个不等价的极大解，二者分别将所有场的依赖范围限制在 7 个坐标和 6 个坐标。它们分别对应十一维超引力和 IIB 型超引力，因此这两种理论都可以嵌入同一个例外场论。具体而言，截面约束 (137) 的解分别基于 $\mathfrak{e}_{7(7)}$ 的分解式 (117) 和 (129，对应根据 (这些嵌入满足方程 (137) 可直接从 $\mathfrak{gl}(1)$ 分次推得) 实现的内坐标 $\partial_m \hookrightarrow \partial_M$ 的嵌入。

$$11\text{D}: \mathfrak{e}_{7(7)} \to \mathfrak{gl}(7)$$

$$56 \to 7_{-3} + 21'_{-1} + 21_{+1} + 7'_{+3},$$

(138)

$$\text{IIB}: \mathfrak{e}_{7(7)} \to \mathfrak{gl}(6) \oplus \mathfrak{sl}(2)$$

$$56 \to 6\,(6,1) - 4\,(6',2) - 2 + (20,1)_0 + (6,2)_{+2} + (6',1)_{+4}.$$

Invariance under local gauge transformations (135) in ExFT is implemented by covariant derivatives

例外场论 (ExFT) 中局域规范变换 (135) 下的不变性由协变导数实现

$$\mathcal{D}_\mu = \partial_\mu - \mathcal{L}_{\mathcal{A}_\mu}, \tag{139}$$

with the vector fields ${\mathcal{A}_\mu}^M$ from (132) transforming as

其中来自 (132) 的矢量场 ${\mathcal{A}_\mu}^M$ 按下式变换

$$\delta_\Lambda {\mathcal{A}_\mu}^M = \partial_\mu \Lambda^M - {\mathcal{A}_\mu}^K \partial_K \Lambda^M + 12 \partial_K {\mathcal{A}_\mu}^L \mathbb{P}^K{}_L{}^M{}_N {\Lambda_\mu}^N - \frac{1}{2} \Lambda^M \partial_K {\mathcal{A}_\mu}^K$$

$$= \mathcal{D}_\mu \Lambda^M. \tag{140}$$

In turn, the gauge covariant field strength is given by

相应地，规范协变场强度由下式给出

$$\mathcal{F}_{\mu\nu}{}^{M} = 2\partial_{[\mu}\mathcal{A}_{\nu]}{}^{M} - 2\mathcal{A}_{[\mu}{}^{K}\partial_{K}\mathcal{A}_{\nu]}{}^{M} - 12(t_{\alpha})^{MK}(t^{\alpha})_{NL}\mathcal{A}_{[\mu}{}^{N}\partial_{K}\mathcal{A}_{\nu]}{}^{L}$$

$$-\frac{1}{2}\Omega^{MK}\mathcal{A}_{[\mu}{}^{N}\partial_{K}\mathcal{A}_{\nu]N} - 12(t^{\alpha})^{MN}\partial_{N}\mathcal{B}_{\mu\nu\alpha} - \frac{1}{2}\Omega^{MN}\mathcal{B}_{\mu\nu N}.$$

(141)

While the non-Abelian part of the field strength immediately follows from the algebra of generalized diffeomorphisms (135), the Stückelberg-type couplings to the 2-forms of (133) are required by gauge covariance since the algebra is a Leibniz rather than a Lie algebra.

虽然场强度的非阿贝尔部分可直接由广义微分同胚代数 (135) 得到，但由于该代数是莱布尼茨代数而非李代数，因此规范协变性要求引入对 (133) 中 2-形式的斯图克尔贝格型耦合。

The dynamics of $E_{7(7)}$ ExFT is described by a twisted self-duality equation for these non-Abelian field strengths, which directly generalizes the corresponding equation of the $D = 4$ theory (122)

$E_{7(7)}$ 例外场论的动力学由这些非阿贝尔场强度的扭曲自对偶方程描述，该方程直接推广了 $D = 4$ 理论 (122) 的对应方程

$$\mathcal{F}_{\mu\nu}{}^{M} = -\frac{1}{2}\,|e|\,\varepsilon_{\mu\nu\rho\sigma}\Omega^{MN}\mathcal{M}_{NK}\mathcal{F}^{\rho\sigma K}. \tag{142}$$

Similar to the $D = 4$ theory, the remaining field equations of $E_{7(7)}$ ExFT are obtained from a pseudo-action whose Lagrangian is directly modeled after (124) as

与 $D = 4$ 理论类似，$E_{7(7)}$ 例外场论的其余场方程由一个伪作用量得到，该伪作用量的拉格朗日量直接仿照式 (124) 构造为

$$\mathcal{L}_{\text{ExFT7}} = |e|\left(\mathcal{R} + \frac{1}{48}g^{\mu\nu}\mathcal{D}_{\mu}\left(\mathcal{M}^{-1}\right)^{MN}\mathcal{D}_{\nu}\mathcal{M}_{MN} - \frac{1}{8}\mathcal{M}_{MN}\mathcal{F}^{\mu\nu M}\mathcal{F}_{\mu\nu}{}^{N}\right)$$

$$+\mathcal{L}_{\text{top}} - |e|\,V(g,\mathcal{M}) \tag{143}$$

upon introducing an internal coordinate dependence for all fields and rendering all terms invariant under the action of generalized diffeomorphisms (135), (140).

这只需要为所有场引入内坐标依赖，并让所有项在广义微分同胚 (135)、(140) 的作用下保持不变。

Here, the Einstein-Hilbert term is constructed from the modified Ricci scalar $\mathcal{R}$, constructed from the external metric $g_{\mu\nu}$ however using covariant derivatives

此处，爱因斯坦-希尔伯特项由修正里奇标量 $\mathcal{R}$ 构造，而修正里奇标量是由外度规 $g_{\mu\nu}$ 借助协变导数构造得到的

$$\mathcal{D}_\mu g_{\upsilon\rho} = \partial_\mu - \mathcal{A}_\mu^K \partial_K g_{\upsilon\rho} - \partial_K \mathcal{A}_\mu^K g_{\upsilon\rho}. \tag{144}$$

Similarly, the scalar kinetic term in (143) is a gauged sigma model with covariant derivatives defined by (139), (135). The Yang-Mills term is built from the field strengths (141), while the non-Abelian topological term is most compactly defined as the boundary contribution of a five-dimensional integral over

类似地，(143) 中的标量动能项是一个规范西塔模型，其协变导数由 (139)、(135) 定义。杨-米尔斯项由场强度 (141) 构造，而非阿贝尔拓扑项最紧凑的定义是将其作为五维积分的边界贡献，积分对

$$d\mathcal{L}_{\text{ top}} \propto \Omega_{MN} \mathcal{F}^M \wedge \mathcal{D}\mathcal{F}^N, \tag{145}$$

with the covariant derivative $\mathcal{D}\mathcal{F}^N$ defined as in (140). Finally, the potential term $V(g, \mathcal{M})$ in (143) is given by

其中协变导数 $\mathcal{D}\mathcal{F}^N$ 按 (140) 中的方式定义。最后，(143) 中的势项 $V(g, \mathcal{M})$ 由下式给出

$$V(g, \mathcal{M}) = -\frac{1}{48} \mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_N \mathcal{M}_{KL} + \frac{1}{2} \mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_L \mathcal{M}_{NK}$$

$$-\frac{1}{4} |e|^{-1} \partial_M |e| \partial_N \mathcal{M}^{MN} - \frac{1}{16} \mathcal{M}^{MN} |e|^{-2} \partial_M |e| \partial_N |e|$$

$$-\frac{1}{4} \mathcal{M}^{MN} \partial_M g^{\mu\upsilon} \partial_N g_{\mu\upsilon} \tag{146}$$

Although not manifest, this term can be shown to be invariant under generalized diffeomorphisms (135) up to total derivatives. It may be thought of as the analogue of a Ricci scalar on the extended internal spacetime [81].

尽管并不明显，但可证明该项在广义微分同胚 (135) 下，除全微分项外保持不变。它可以被视为扩展内空间上里奇标量的类比 [81]。

By construction, all terms in (143) are separately invariant under generalized diffeomorphisms (135), (140). Moreover, the relative coefficients among these terms are uniquely fixed by further demanding invariance under properly defined external diffeomorphisms. The latter are generated by vectors ξ^μ, which however also depend on the internal coordinates. Invariance under generalized internal and external diffeomorphisms thus uniquely fixes the dynamics of $E_{7(7)}$ ExFT.

通过构造可知，(143) 中的所有项都分别在广义微分同胚 (135)、(140) 下不变。此外，这些项之间的相对系数由恰当定义的外微分同胚不变性要求唯一确定。外微分同胚由矢量 ξ^μ 生成，不过 ξ^μ 也依赖于内坐标。因此，广义内微分同胚与外微分同胚下的不变性唯一确定了 $E_{7(7)}$ 例外场论 (ExFT) 的动力学。

After picking a solution (139) of the section constraint, the field equations obtained from (142) and (143) precisely reproduce the field equations of 11D and IIB supergravity, respectively. Moreover, also massive IIA supergravity, c.f. (29), can be reproduced upon further deformation of the gauge structures [87, 88]. All these theories are thus united within a common framework. Moreover, after dimensional reduction $\partial_M \to 0$, this formulation directly yields the $D=4$ theory (122),(124), with the global symmetry group $E_{7(7)}$ manifest. This makes ExFT a natural framework for the study of duality and solution generating transformations (see, e.g., [89-91]). Moreover, with the higher-dimensional fields already rearranged such as to fit the fields of the lower-dimensional theory, $E_{7(7)}$ ExFT (143) is precisely tailored for the study of reductions to $D=4$ dimensions. It has been a particularly powerful tool for the construction of consistent trunctions [92, 93] and the computation of Kaluza-Klein spectra around four-dimensional backgrounds [94].

选取截面约束的一个解 (139) 后，由 (142) 和 (143) 得到的场方程恰好分别重现了 11 维超引力和 IIB 型超引力的场方程。此外，有质量 IIA 型超引力 (参见 (29)) 也可以通过对规范结构进一步变形后得到 [87, 88]。因此所有这些理论都被统一到同一个框架中。此外，经过维约化 $\partial_M \to 0$ 后，该表述直接给出了理论 (122)、(124) $D=4$，且整体对称群 $E_{7(7)}$ 是显式的。这使得例外场论成为研究对偶性和解生成变换的自然框架 (例如参见 [89-91])。此外，由于高维场已经预先整理为适配低维理论场的形式，(143) 的 $E_{7(7)}$ 例外场论恰好适合研究约化到 $D=4$ 维的情形。它一直是构造相容截断 [92, 93] 和计算四维背景下卡鲁扎-克莱因能谱 [94] 的特别有力的工具。

Exceptional field theories have been constructed for all finite-dimensional duality groups $E_{d(d)}$ (i.e., for $d \leq 8$) $[21, 36, 75, 95-99]$. Just as (143), the respective actions are modeled after the structure of the $(11-d)$ -dimensional maximal supergravities, lifting all fields to an extended spacetime (subject to the section contraint), with the non-Abelian gauge structure induced by the infinite-dimensional algebraic structure of generalized diffeomorphisms. For $d > 8$, the exceptional field theory based on the infinite-dimensional affine algebra $\mathfrak{e}_{9(9)}$ has been constructed in [100,101]. Extrapolating the structures all the way to the very-extended Kac-Moody algebra $\mathfrak{e}_{11}$, a master formulation has been given in [102]. This also allows to make contact with the E_{11} conjectures of [55, 103, 104] and the E_{10} conjecture of [74].

目前已经针对所有有限维对偶群 $E_{d(d)}$ 构造了例外场论 (即对于 $d \leq 8$) $[21, 36, 75, 95-99]$ 。与 (143) 一样，各对应作用量都仿照 $(11-d)$ 维最大超引力的结构构建，将所有场提升到扩展时空 (满足截面约束)，非阿贝尔规范结构由广义微分同胚的无限维代数结构诱导。对于 $d > 8$ ，基于无限维仿射代数 $\mathfrak{e}_{9(9)}$ 的例外场论已在 [100,101] 中构造完成。通过将结构一路外推到超扩展卡茨-穆迪代数 $\mathfrak{e}_{11}$ ，文献 [102] 给出了一个主表述。这也使得该理论可以和 E_{11} 的 [55, 103, 104] 猜想以及文献 [74] 的 E_{10} 猜想建立联系。

Let us finally note that although we have restricted here to a discussion of the bosonic sector, the ExFT construction can be extended to the fermionic sector in a unique way such that supersymmetry can be realized [83, 105-108]. It is however interesting to note that, as stated above, in this framework, the bosonic sector is already uniquely determined by imposing purely bosonic symmetries, the generalized internal and external diffeomorphisms. This of course is directly related to the observation discussed after (80) that the appearance of hidden symmetries relies on the exact bosonic couplings determined by supersymmetry. Why supersymmetry precisely selects the couplings that give rise to the symmetry enhancement and the exceptional groups still remains somewhat of a mystery and continues to challenge our understanding of the fundamental symmetries of supergravity.

最后我们要指出，尽管本文仅讨论玻色子部分，例外场论的构造可以唯一方式扩展到费米子部分，从而实现超对称性 [83, 105-108]。但有意思的一点是，正如上文所述，在该框架下，仅要求纯玻色对称性 (广义内微分同胚与外微分同胚) 就可以唯一确定玻色子部分。这当然直接关联到 (80) 之后讨论的结论: 隐藏对称性的出现依赖于超对称性确定的精确玻色耦合。超对称性为何恰好选出能带来对称性增强和例外群的耦合，这一点至今仍存在谜团，也持续挑战着我们对超引力基本对称性的理解。

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